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Evaluation of Reliability and Availability Characteristics of Two Different Systems by Using Linear First Order Differential Equations

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Abstract: This study deals with the reliability and availability characteristics of two different series system configurations. The second system differs from the first system due to the additional feature of preventive maintenance. The failure times of a components are assumed to be exponentially distributed with parameters α, β, α' and β', respectively. The replacement time distribution of each server is also exponentially distributed with parameters γ, δ and θ. We derive the mean time-to-failure, *MTTF*, and the steady-state availability, $A_T(\infty)$, for the two systems using linear first order differential equations and perform comparisons theoretically and graphically to observe the effect of the preventive maintenance on system performance.

Key words: Reliability and Availability Characteristics, Preventive Maintenance, System Performance

INTRODUCTION

Several authors [1, 3] studied a two similar unit standby redundant system with preventive maintenance, inspection and two types of repair. Goel and Shrivastava [4] studied the reliability and availability of two systems, the second system differs from the first system due to the additional feature of preventive maintenance. Researchers in reliability have shown a keen interest in the analysis of two (or more) component parallel systems owing to their practical utility in modern industrial and technological set-ups. Of these systems, more commonly used are those in which the failure in one component affects the failure rates of other components. For example we can consider engine failure in two engine planes, wear of two pens on an executives desk or the performance of an individuals eyes, ears, kidneys and other paired physical organs. The present paper is devoted to deal with two systems, each system with two dissimilar components arranged in parallel, the second system differ from the first system due to the additional feature of preventive maintenance at random epochs. The mean time-to-failure, *MTTF*, and the steady-state availability, $A_T(\infty)$, for the two systems are obtained using a novel methodology (Linear First Order Differential Equations) [5]. The effect of preventive maintenance on the system performance is shown by performing comparisons theoretically and graphically. The following assumptions are common for the two systems

* The system consists of a single unit having two dissimilar parallel Components, say A and B.
* The system remains operative even if a single component operates.
* The failure of a component changes the life time parameter of the other.
* Upon failure $\rightarrow$ each component can be replaced with a similar component with both the component (when failed) can also be replaced simultaneously
* After replacement of each component , the system is as good as new.

But in the second system we assume that

* Preventive maintenance (e.g. overhaul, inspection, minor repairs, etc.) is provided to this system at random epochs when the system is in the state S_0. Where both the components are normal.

The common symbols of the two systems are:

α	Unconditional failure rate of component A
β	Unconditional failure rate of component B
α'	Failure rate of component A when B has already failed
β'	Failure rate of component B when A has already failed
γ	Replacement rate of component A
δ	Replacement rate of component B
θ	Replacement rate of component A , B
$P_{n,i}(t)$	Probability that exactly n components are working at time t, $(t \geq 0)$ at state S_i
$u(t)$	pdf of time for taking a unit into preventive maintenance i.e., $u(t) = \lambda \exp(-\lambda t), \lambda, t > 0$

$v(t)$ pdf and of preventive maintenance time, $v(t) = \mu \exp(-\mu t), \mu, t > 0$

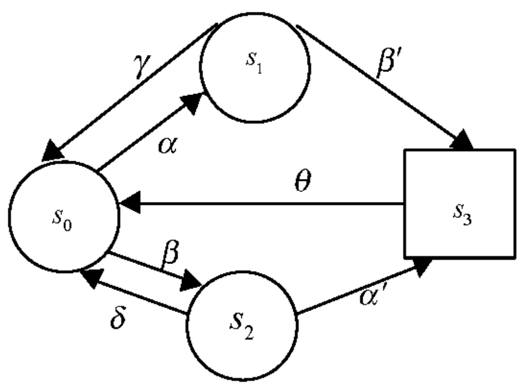
A unit can be in one of the following states at time t

- A_N Component A in normal mode and operative
- B_N Component B in normal mode and operative
- A_F Component A in failure mode and needs replacement
- B_F Component B in failure mode and needs replacement
- A_{NP} Component A in normal mode and under preventive maintenance
- B_{Np} Component B in normal mode and under preventive maintenance

The First System

(a) Up States: $S_0 = (A_N, B_N)$, $S_1 = (A_F, B_N)$, $S_2 = (A_N, B_F)$

(b) Down States : $S_3 = (A_F, B_F)$



Up state
Down State

Fig. 1: State Transition Diagram for the First System

Mean Time to System Failure: For Fig.1, let $P_{n,i}(t)$ Probability that exactly n components are working at time t, $(t \geq 0)$ at state S_i. If we let $P(t)$ denote the probability row vector at time t, then the initial conditions for this problem are

$$P(0) = [P_{2,0}(0), P_{1,1}(0), P_{1,2}(0), P_{0,3}(0)] = [1, 0, 0, 0] \quad (1)$$

We obtain the following differential equation:

$$\frac{dP_{2,0}}{dt} = -(\alpha + \beta)P_{2,0} + \gamma P_{1,1} + \delta P_{1,2} + \theta P_{0,3}$$

$$\frac{dP_{1,1}}{dt} = \alpha P_{2,0} - (\gamma + \beta')P_{1,1}$$

$$\frac{dP_{1,2}}{dt} = \beta P_{2,0} - (\delta + \alpha')P_{1,2}$$

$$\frac{dP_{0,3}}{dt} = \beta' P_{1,1} + \alpha' P_{1,2} - \theta P_{0,3}$$

this can be written in the matrix form as

$$\dot{P} = QP, \quad (2)$$

where:

$$Q = \begin{bmatrix} -(\alpha+\beta) & \gamma & \delta & \theta \\ \alpha & -(\gamma+\beta') & 0 & 0 \\ \beta & 0 & -(\delta+\alpha') & 0 \\ 0 & \beta' & \alpha' & -\theta \end{bmatrix}$$

To evaluate the transient solution is too complex. Therefore, we will restrict ourselves in calculating the $MTTF_1$. To calculate the $MTTF_1$, we take the transpose matrix of Q and delete the rows and columns for the absorbing states. The new matrix is called A. The expected time to reach an absorbing state is calculated from

$$E[T_{P(0)\to P(absorbing)}] = P(0)(-A^{-1})\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (3)$$

where:

$$A = \begin{pmatrix} -(\alpha+\beta) & \alpha & \beta \\ \gamma & -(\gamma+\beta') & 0 \\ \delta & 0 & -(\delta+\alpha') \end{pmatrix}$$

This method is successful of the following relations:

$$E[T_{P(0)\to P(absorbing)}] = P(0)\int_0^{\infty} e^{At}\,dt \quad (4)$$

and

$$\int_0^{\infty} e^{At}\,dt = -A^{-1} \quad (5)$$

We obtain the following explicit expression for the $MTTF_1$

$$E_1[T_{P(0)\to P(absorbing)}] = MTTF_1 = \frac{(\gamma+\beta')(\delta+\alpha') + \alpha(\alpha'+\delta) + \beta(\beta'+\gamma)}{[\alpha\beta'(\delta+\alpha') + \beta\alpha'(\gamma+\beta')]}$$

Availability Analysis of the System: For the availability case of Fig.1, the initial conditions for this problem are the same as for the reliability case:

$$P(0) = [P_{2,0}(0), P_{1,1}(0), P_{1,2}(0), P_{0,3}(0)] = [1,0,0,0] \quad (7)$$

The differential equations form can be expressed as:

$$\begin{pmatrix} \dot{P}_{2,0} \\ \dot{P}_{1,1} \\ \dot{P}_{1,2} \\ \dot{P}_{0,3} \end{pmatrix} = \begin{pmatrix} -(\alpha+\beta) & \gamma & \delta & \theta \\ \alpha & -(\beta'+\gamma) & 0 & 0 \\ \beta & 0 & -(\alpha'+\delta) & 0 \\ 0 & \beta' & \alpha' & -\theta \end{pmatrix} \begin{pmatrix} P_{2,0} \\ P_{1,1} \\ P_{1,2} \\ P_{0,3} \end{pmatrix}$$

The steady-state availabilities can be obtained using the following procedure. In the steady-state, the derivatives of the state probabilities become zero. That allows us to calculate the steady-state probabilities with.

$$A_{T_1}(\infty) = 1 - P_{0,3}(\infty) \quad (8)$$

and
$QP(\infty) = 0,$

or, in the matrix form:

$$\begin{pmatrix} -(\alpha+\beta) & \gamma & \delta & \theta \\ \alpha & -(\beta'+\gamma) & 0 & 0 \\ \beta & 0 & -(\alpha'+\delta) & 0 \\ 0 & \beta' & \alpha' & -\theta \end{pmatrix} \begin{pmatrix} P_{2,0}(\infty) \\ P_{1,1}(\infty) \\ P_{1,2}(\infty) \\ P_{0,3}(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (9)$$

To obtain $P_{0,3}(\infty)$, we solve (9) and the following normalizing condition:

$$P_{2,0}(\infty) + P_{1,1}(\infty) + P_{1,2}(\infty) + P_{0,3}(\infty) = 1 \quad (10)$$

we substitute (10) in any one of the redundant rows in (9) to yield

$$\begin{pmatrix} -(\alpha+\beta) & \gamma & \delta & \theta \\ \alpha & -(\beta'+\gamma) & 0 & 0 \\ \beta & 0 & -(\alpha'+\delta) & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} P_{2,0}(\infty) \\ P_{1,1}(\infty) \\ P_{1,2}(\infty) \\ P_{0,3}(\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$
(11)

The solution of (11) provides the steady-state probabilities in the availability case. For Fig.1, the explicit expression for $A_{T_1}(\infty)$ is given by:

$$A_{T_1}(\infty) = \frac{\theta\{\alpha(\alpha'+\delta) + \beta(\beta'+\gamma) + (\beta'+\gamma)(\alpha'+\delta)\}}{\theta\{\alpha(\alpha'+\delta) + \beta(\beta'+\gamma) + (\beta'+\gamma)(\alpha'+\delta)\} + \alpha\beta'(\alpha'+\delta) + \beta\alpha'(\beta'+\gamma)} \quad (12)$$

The Second System

Up States: $S_0 = (A_N, B_N)$, $S_1 = (A_F, B_N)$, $S_2 = (A_N, B_F)$, $S_4 = (A_{Np}, B_{Np})$

Down States: $S_4 = (A_F, B_F)$

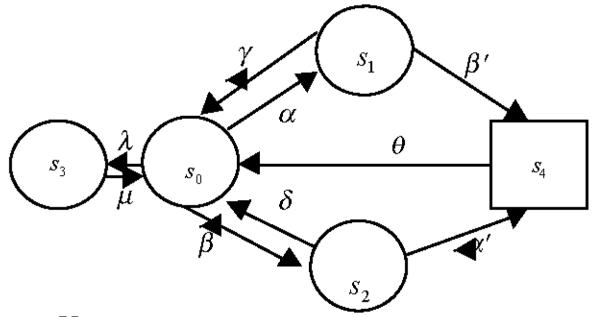


Fig. 2: State Transition Diagram for the Second System

Mean Time to System Failure: For Fig. 2, let $P_{n,i}(t)$ Probability that exactly n components are working at time t, $(t \geq 0)$ at state S_i. If we let $P(t)$ denote the probability row vector at time t, then the initial conditions for this problem are :

$$P(0) = [P_{2,0}(0), P_{1,1}(0), P_{1,2}(0), P_{2,3}(0), P_{0,4}] = [1, 0, 0, 0, 0] \quad (13)$$

We obtain the following differential equation:

$$\frac{dP_{2,0}}{dt} = -(\alpha+\beta+\lambda)P_{2,0} + \gamma P_{1,1} + \delta P_{1,2} + \mu P_{2,3} + \theta P_{0,4}$$
$$\frac{dP_{1,1}}{dt} = \alpha P_{2,0} - (\gamma+\beta')P_{1,1}$$
$$\frac{dP_{1,2}}{dt} = \beta P_{2,0} - (\delta+\alpha')P_{1,2}$$
$$\frac{dP_{2,3}}{dt} = \lambda P_{2,0} - \mu P_{2,3}$$
$$\frac{dP_{0,4}}{dt} = \beta' P_{1,1} + \alpha' P_{1,2} - \theta P_{0,4}$$

this can be written in the matrix form as:

$$\dot{P} = QP, \quad (14)$$

where :

$$Q = \begin{pmatrix} -(\alpha+\beta+\lambda) & \gamma & \delta & \mu & \theta \\ \alpha & -(\gamma+\beta') & 0 & 0 & 0 \\ \beta & 0 & -(\delta+\alpha') & 0 & 0 \\ \lambda & 0 & 0 & -\mu & 0 \\ 0 & \beta' & \alpha' & 0 & -\theta \end{pmatrix}$$

To evaluate the transient solution is too complex. Therefore, we will restrict ourselves in calculating the $MTTF_2$. To calculate the $MTTF_2$, we take the transpose matrix of Q and delete the rows and columns for the absorbing states. The new matrix is called A. The expected time to reach an absorbing state is calculated from

$$E\left[T_{P(0)\to P(absorbing)}\right] = P(0)(-A^{-1})\begin{pmatrix}1\\1\\1\end{pmatrix}, \tag{15}$$

where

$$A = \begin{pmatrix} -(\alpha+\beta+\lambda) & \alpha & \beta & \lambda \\ \gamma & -(\gamma+\beta') & 0 & 0 \\ \delta & 0 & -(\delta+\alpha') & 0 \\ \mu & 0 & 0 & -\mu \end{pmatrix}$$

This method is successful of the following relations:

$$E\left[T_{P(0)\to P(absorbing)}\right] = P(0)\int_0^\infty e^{At}\,dt \tag{16}$$

and

$$\int_0^\infty e^{At}\,dt = -A^{-1} \tag{17}$$

We obtain the following explicit expression for the $MTTF_2$:

$$E_2\left[T_{P(0)\to P(absorbing)}\right] = MTTF_2 = \frac{(\mu+\lambda)(\gamma+\beta')(\delta+\alpha') + \mu\alpha(\alpha'+\delta) + \mu\beta(\beta'+\gamma)}{\mu[\alpha\beta'(\delta+\alpha') + \beta\alpha'(\gamma+\beta')]} \tag{18}$$

Availability analysis: For the availability case of Fig. 2, the initial conditions are the same as for the reliability case

$$P(0) = \left[P_{2,0}(0), P_{1,1}(0), P_{1,2}(0), P_{2,3}(0), P_{0,4}(0)\right] = [1,0,0,0,0] \tag{19}$$

The differential equations form can be expressed as:

$$\begin{pmatrix} \dot P_{2,0} \\ \dot P_{1,1} \\ \dot P_{1,2} \\ \dot P_{2,3} \\ \dot P_{0,4} \end{pmatrix} = \begin{pmatrix} -(\alpha+\beta+\lambda) & \gamma & \delta & \mu & \theta \\ \alpha & -(\gamma+\beta') & 0 & 0 & 0 \\ \beta & 0 & -(\delta+\alpha') & 0 & 0 \\ \lambda & 0 & 0 & -\mu & 0 \\ 0 & \beta' & \alpha' & 0 & -\theta \end{pmatrix} \begin{pmatrix} P_{2,0} \\ P_{1,1} \\ P_{1,2} \\ P_{2,3} \\ P_{0,4} \end{pmatrix}$$

The steady-state availabilities can be obtained using the following procedure. In the steady-state, the derivatives of the state probabilities become zero. That allows us to calculate the steady-state probabilities with:

$$A_{T_1}(\infty) = 1 - P_{0,4}(\infty) \tag{20}$$

and

$$QP(\infty) = 0, \tag{21}$$

or, in the matrix form

$$\begin{pmatrix} -(\alpha+\beta+\lambda) & \gamma & \delta & \mu & \theta \\ \alpha & -(\gamma+\beta') & 0 & 0 & 0 \\ \beta & 0 & -(\delta+\alpha') & 0 & 0 \\ \lambda & 0 & 0 & -\mu & 0 \\ 0 & \beta' & \alpha' & 0 & -\theta \end{pmatrix} \begin{pmatrix} P_{2,0}(\infty) \\ P_{1,1}(\infty) \\ P_{1,2}(\infty) \\ P_{2,3}(\infty) \\ P_{0,4}(\infty) \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix} \tag{22}$$

To obtain $P_0(\infty)$, we solve (22) and the following normalizing condition:

$$P_{2,0}(\infty) + P_{1,1}(\infty) + P_{1,2}(\infty) + P_{2,3}(\infty) + P_{0,4}(\infty) = 1 \tag{23}$$

we substitute (23) in any one of the redundant rows in (22) to yield :

$$\begin{pmatrix} -(\alpha+\beta+\lambda) & \gamma & \delta & \mu & \theta \\ \alpha & -(\gamma+\beta') & 0 & 0 & 0 \\ \beta & 0 & -(\delta+\alpha') & 0 & 0 \\ \lambda & 0 & 0 & -\mu & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} P_{2,0}(\infty) \\ P_{1,1}(\infty) \\ P_{1,2}(\infty) \\ P_{2,3}(\infty) \\ P_{0,4}(\infty) \end{pmatrix} = \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix} \tag{24}$$

The solution of (24) provides the steady-state probabilities in the availability case. For Fig. 2, the explicit expression for $A_{T_2}(\infty)$ is given by:

$$A_{T_2}(\infty) = \frac{\theta\{\{\mu\alpha + (\mu+\lambda)(\beta'+\gamma)\}(\alpha'+\delta) + \beta\mu(\beta'+\gamma)\}}{\theta\{\{\alpha\mu + (\mu+\lambda)(\beta'+\gamma)\}(\alpha'+\delta) + \beta\mu(\beta'+\gamma)\} + \mu\{\alpha\beta'(\alpha'+\delta) + \beta\alpha'(\beta'+\gamma)\}} \tag{25}$$

Numerical Example: To observe the effect of preventive maintenance on system behavior, we plot the *MTTF* and steady –state availability for the models, against α, β, respectively keeping the other parameters fixed at $\alpha' = 2.0$, $\beta' = 2.5$, $\delta = 2.0$, $\theta = 3.0$, $\lambda = 2.8$, $\mu = 4.0$. For curves against α we take $\beta = 0.6$ and in the curves against β we take $\alpha = 0.8$ in addition to the above parameters.

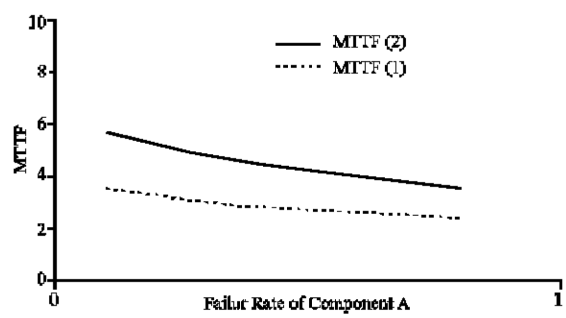


Fig. 3: Comparing of MTTF w.r.t. Failure of Component A

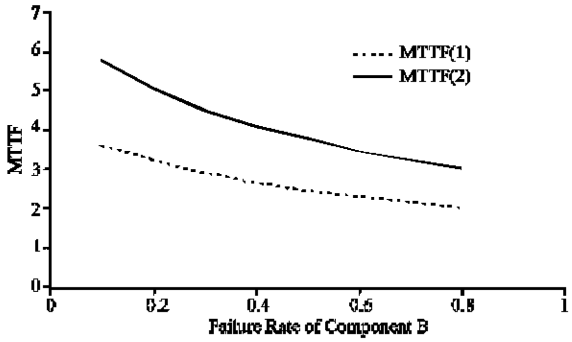


Fig. 4: Comparing of MTTF w.r.t. Failure Rate of Component B

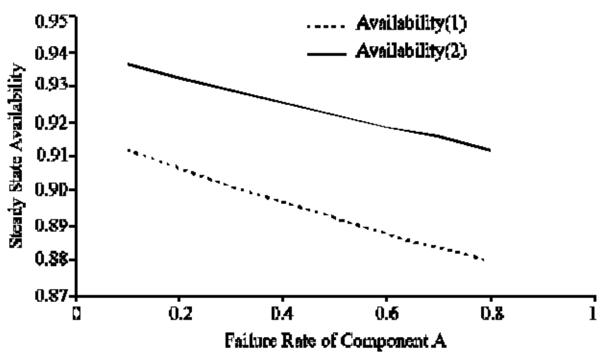


Fig. 5: Comparing of Steady State Availability w.r.t. Failure Rate of Component A

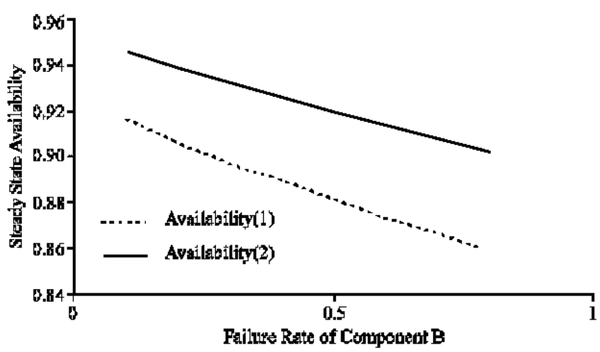


Fig. 6: Comparing of Steady State Availability w.r.t. Failure Rate of Component B

CONCLUSION

We use computer software, to compare two configurations in terms of their $MTTF_i$ and their $A_{T_i}(\infty)$, where $i = 1,2$. We first perform a comparison for the $MTTF$ showed in Fig. 3 and 4, we get the following results:

$$MTTF_2 > MTTF_1$$

Next, the comparison of $A_T(\infty)$ showed in Fig. 5 and 6, we get the following results:

$$A_{T_2} > A_{T_1}$$

under general assumption, graphically and from the comparison of two systems we get. The second system is briefer than the first system when additional preventive maintenance.

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