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Strong Convergence Theorems for Fixed Points of Asymptotically Nonexpansive Mappings in Uniformly Smooth Banach Spaces

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Abstract: In this work, we establish strong convergence of the improved modified three-step iteration methods for asymptotically nonexpansive operators in uniformly smooth and convex Banach spaces. Our results generalize the previous results of Schu [10], Osilike and Aniagbosor [5], Xu and Noor [13], as well as Owojori and Imoru [7] among others.

Key words: Asymptotically Nonexpansive Mappings; Uniformly Smooth Banach Spaces; Three-Step Iteration Methods

INTRODUCTION

An operator $T:K\rightarrow K$, where K is a nonempty subset of a Banach space, is called nonexpansive if for all $x,y \in$

K we have $T_x - T_y \leq x-y$ (1.1)

holds for all $x,y \in K$, where s_1, s_2, s_3 are real constants in [0,1] satisfying
$s_1 + 2s_2 + 2s_3 = 1$.
Goebel and Kirk [2] introduced the concept of asymptotically nonexpansive operators as a generalization of nonexpansive mappings. An operator T is called asymptotically nonexpansive if there exists a real sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$. such that

$$T^n x - T^n{}_y \leq k_n \ x-y \ , \forall\, n \in N \qquad (1.2)$$

Goebel and Kirk [2] introduced the concept of asymptotically nonexpansive selfmapping of a uniformly smooth and convex Banach space has a fixed point.
Schu [10] introduced the modified Mann and Ishikawa iteration methods for arbitrary $x_1 \in K$ by

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, \qquad n \geq 1 \qquad (1.3)$$

and respectively

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n x_n + c_n u_n, \\ y_n &= a_n' x_n + b_n' T^n x_n + c_n' v_n, \end{aligned}\right\} \quad n \geq 1 \qquad (1.4)$$

where$\{u_n\}$, $\{v_n\}$ are bounded sequences in K-a close bounded convex subset of uniformly smooth Banach space and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$ $\{c_n'\}$, are sequences in [0,1] satisfying:
$a_n + b_n + c_n = a_n' + b_n' = 1$
for all $n \geq 0$.

When $b_n' = c_n' = 0$, for all $n \geq 0$, then (1.4) reduces to (1.3)-the modified Mann iteration scheme with errors in the sense of Xu [15].
These iteration methods have been investigated by several authors including Schu [10], Osilike and Igbokwe [6], Qihou [9], Osilike and Aniagbosor [5], Tan and Xu [11], Goebel and Kirk [2]. Lim and Xu [3] as well as Chidume and Osilike [1] among others.
Xu and Noor[13] introduced and analyzed the following modified three-step iteration scheme for asymptotically nonexpansive mappings in Banach spaces:

Definition 1.1: Let D be a nonempty subset of a normed space B and T: $D\rightarrow D$ be a mapping. For a given $x_o \in D$, compute sequences $\{x_n\}$, $\{y_n$, $\{z_n$ by the iteration scheme

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n z_n \qquad n \geq 0 \qquad (1.5) \\ z_n &= (1-\gamma_n)x_n + \gamma_n T^n x_n \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are arbitrary sequences in [0,1].
We observe that when $\gamma_n = 0$, then (1.5) reduces to

$$\left.\begin{aligned} xn+1 &= (1-\ n)xn +\ nTnyn \\ yn &= (1-\ n)xn +\ nTnxn \end{aligned}\right\} n \geq 0$$

which is the modified Ishikawa iteration scheme, (without errors). In addition, when $\beta_n = \gamma_n = 0$, then (1.5) reduces to the modified Mann iteration scheme given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \ n \geq 0$$

Xu and Noor [13] then established the convergence of (1.5) to the fixed point of asymptotically nonexpansive operator in uniformly smooth Banach space.

Owojori and Imoru [7] also introduced a modified three-step iteration scheme with errors which include that of Xu and Noor [13] as a special case and then established a strong convergence result.

Recently, an improved three-step method with errors was introduced by Owojori and Imoru [8] with errors as a generalization of the Mann and Ishikawa iteration methods with errors. Convergence of the improved iteration method was established for pseudocontractive and accretive operations in arbitrary Banach spaces.

An improved modified three-step iteration scheme with errors which includes previous modified iteration methods as special cases is given by the following:

Definition 1.1: Let K be a nonempty closed bounded convex subset of a uniformly smooth Banach space and suppose T,S are uniformly continuous asymptotically nonexpansive selfmappings of K. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n S^n x_n \\ y_n &= a'_n x_n + b'_n S^n z_n + c'_n v_n \\ z_n &= a''_n x_n + b''_n T^n x_n + c''_n w_n \end{aligned}\right\} \quad n \geq 1 \tag{1.6}$$

where $\{v_n\}$, $\{w_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$, $\{b''_n\}$, $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$, are real sequences in $[0, 1]$ satisfying the following conditions:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$,

(ii) $\sum b_n = \infty$,

(iii) $\alpha_n := b_n + c_n$, $\beta_n := b'_n + c'_n$, $\Upsilon_n := b''_n + c''_n$

Remark: A special case of the modified iteration scheme (1.6) is also given by:

$$\left.\begin{aligned} xn+1 &= anxn + bnTnyn + cnun \\ yn &= a'_n x_n + b'_n T^n z_n + c'_n v_n \\ zn &= a''_n x_n + b''_n T^n x_n + c''_n w_n \end{aligned}\right\} n \geq 1 \tag{1.7}$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K and $\{a_n\}$, $\{a'_n\}$, $\{a''_n\}$, $\{b_n\}$, $\{b'_n\}$, $\{b''_n\}$, $\{b''_n\}$, $\{c_n\}$, $\{c'_n\}$, $\{c''_n\}$, are real sequences in $[0, 1]$ satisfying

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$,

(ii) $\sum b_n = \infty$

We readily observe that the revised iteration methods (1.6) and (1.7) are extensions of the modified three-step iteration scheme of B. Xu and Noor [13], with $\alpha_n := b_n$, $\beta_n := b'_n$, $\Upsilon_n := b''_n$, as well as the modified Mann and Ishikawa iteration methods with (and without) errors in the sense of Liu [4] and Xu [15].

Some Preliminary Results: In the sequel, we shall require the following results.

Lemma 2.1 (Xu [14]): Let B be a uniformly smooth Banach space. Then B has modulus of smoothness of power type $q > 1$ if and only if there exist $j_q x \in J_q x$ and a constant $c > 0$ such that

$$\|x + y\|^q \leq \|x\|^q + q < y, j_q(x) > + c\|y\|^q \tag{2.1}$$

for all $x, y \in B$.

By replacing y with (-y) in (2.1), we obtain

$$\|x - y\|^q \leq \|x\|^q - q < y, j(x) > + \|y\|^q \leq \|x\|^q + \|y\|^q \tag{2.2}$$

for all $x, y \in B$.

Applying Lemma 2.1, Chidume and Osilike [1] established the following result in real uniformly smooth Banach spaces with modulus of smoothness of power type $q > 1$.

Lemma 2.2: Let B be a uniformly smooth Banach space with modulus of smoothness of power type $q > 1$. Then for all $x,y,z \in B$ and $\lambda \in [0,1]$, the following inequality

$$\|\lambda x + (1 - \lambda)y - z\|^q \leq [1 - \lambda(q - 1)]\|y - z\|^q + \lambda c\|x - z\|^q - \lambda[1 - \lambda^{q-1}c]\|x - y\|^q \qquad (2.3)$$

holds, where c is a positive constant.

The following result due to Weng [12] is also useful in this study.

Lemma 2.3: Let $\{\Phi_n\}$ be a nonnegative sequence of real numbers satisfying

$$\Phi_{n+1} \leq (1 - \delta_n)\Phi_n + \sigma_n \qquad (2.4)$$

where $\delta_n \in [0,1]$, $\sum\delta_n = \infty$ and $\sigma_n = o(\delta_n)$. Then $\lim_{n\to\infty}\Phi_n = 0.$

RESULTS

Our main result is the following.

Theorem 3.1: - Let B be a uniformly smooth Banach space and K a nonempty closed bounded and convex subset of B. Suppose T is a completely continuous asymptotically nonexpansive selfmapping of K with real sequence $\{k_n\}$ satisfying, :$k_n \geq 1$, $\forall n$, with $\lim_{n\to\infty} k_n = 1.$ and $k_n^p + 1 \leq p$, $p > 1$. For a given $x_1 \in K$, define sequence $\{x_n\}$ generated iteratively by

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n \\ y_n &= a_n' x_n + b_n' T^n z_n + c_n' v_n \\ z_n &= a_n'' x_n + b_n''' T^n x_n + c_n'' w_n \end{aligned}\right\} \quad n \geq 1$$

where $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n'\}$, $\{a_n''\}$, $\{b_n\}$, $\{b_n'\}$, $\{b_n''\}$, $\{b_n''\}$, $\{c_n\}$, $\{c_n'\}$, $\{c_n''\}$ are real sequences in [0,1] satisfying the following conditions

(i) $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$,

(ii) $\sum b_n = \infty$,

(iii) $\alpha_n := b_n + c_n \beta_n := a_n' + b_n' + c_n' \Upsilon_n := b_n'' + c_n''$,

(iv) $(p - 1 - k_n^p) \leq \dfrac{1}{1 + \beta_n k_n^p(1 + \gamma_n k_n^p)}$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof:- By Goebel and Kirk[2], T has a fixed point in K.

Let $x^* \in K$ be a fixed point of T. Then, from our hypothesis and Lemma 2.2, we have the following estimates.

$$\begin{aligned} \|z_n - x^*\|^p &= \|a_n'' x_n + b_n''' T^n x_n + c_n'' w_n - x^*\|^p \\ &= \|[(1 - \gamma_n)(x_n - x^*) + \gamma_n(T^n x_n - x^*)] - c_n''(T^n x_n - w_n)\|^p \\ &\leq [1 - \gamma_n(p - 1)]\|(x_n - x^*) - c_n''(T^n x_n - w_n)\|^p \\ &\quad + \gamma_n c\|(T^n x_n - x^*) - c_n''(T^n x_n - w_n)\|^p \\ &\quad - \gamma_n(1 - \gamma_n^{p-1}c)\|(T^n x_n - x^*) - (x_n - x^*)\|^p \end{aligned}$$

Observe that $\gamma_n(1 - \gamma_n^{p-1}c) \geq 0$. Therefore, expanding further, we obtain

$$\begin{aligned} \|z_n - x^*\|^p &\leq [1 - \gamma_n(p - 1)][\|x_n - x^*\|^p + cc_n''\|T^n x_n - w_n\|^p \\ &\quad - p < c_n''(T^n x_n - w_n), j(x_n - x^*) >] + \gamma_n c[\|T^n x_n - x^*\|^p + cc_n''\|T^n x_n - w_n)\|^p \end{aligned}$$

$$- p < c''_n (T^n x_n - w_n), j(x_n - x^*) >] \quad (3.1)$$

$$\leq [1-\gamma_n (p-1)][\| x_n - x^* \|^p + cc''_n \| T^n x_n - w_n \|^p]$$

$$+ \gamma_n c[\| T^n x_n - x^* \|^p + cc''_n \| T^n x_n - w_n) \|^p]$$

Continuity of T on the bounded set K implies that there exists a real number $N_4 < \infty$ such that $\| T^n x_n - w_n) \|^p \leq N_4$. Observe that $cc''_n < 1$ and T is asymptotically nonexpansive. Therefore,

$$\| z_n - x^* \|^p \leq [1-\gamma_n (p-1)][\| x_n - x^* \|^p + \gamma_n \| T^n x_n - x^* \|^p + [1-\gamma_n (p-1) + \gamma_n] N_4$$

$$\leq [1 - \gamma_n (p - k_n^p - 1)]] \| x_n - x^* \|^p + \{[1-\gamma_n (p-1)] + \gamma_n\} N_4$$

$$= [1 - \gamma_n (p - k_n^p - 1)] \| x_n - x^* \|^p + [1-\gamma_n (p-2)] N_4 \quad (3.2)$$

We also have the following estimates:

$$\|(y_n - x^* \|^p = \| a'_n x_n + b'_n T z_n + c''_n v_n - x^* \|^p$$

$$= \| [1-\beta_n)(x_n - x^*) + \beta_n (T^n z_n - x^*)] - c'_n (T^n z_n - v_n) \|^p$$

$$\leq [1 - \beta_n (p-1)] \| (x_n - x^*) - c'_n (T^n z_n - v_n) \|^p$$

$$+ \beta_n c \| (T^n z_n - x^*) - c'_n (T^n z_n - v_n) \|^p$$

$$- \beta_n (1 - \beta_n^{p-1} c) \| (T^n z_n - x^*) - (x_n - x^*) \|^p$$

Expanding further and considering the fact that $\mathbf{c}, c'_n \leq 1$ and $\beta_n (1 - \beta_n^{p-1} c) \geq 0$,

we have

$$\|(y_n - x^* \|^p \leq [1-\beta_n (p-1)]\{ \| (x_n - x^*) \|^p + \| (T^n z_n - v_n) \|^p$$

$$- p < c_n (T^n z_n - v_n), j(x_n - x^*) > \}$$

$$+ \beta_n \{ \| T^n z_n - x^* \|^p + \| T^n z_n - v_n \|^p$$

$$- p < c_n (T^n z_n - v_n), j(T^n z_n - x^*) > \}$$

$$\leq [1-\beta_n (p-1)]\{ \| x_n - x^* \|^p + \| T^n z_n - v_n \|^p\}$$

$$+ \beta_n \{ \| T^n z_n - x^* \|^p + \| T^n z_n - v_n \|^p\} \quad (3.4)$$

Since T is continuous on K, then there exists a real number N5 < ∞ such that $\| T^n z_n - x^* \|^p \leq N_5$. Therefore

$$\| y_n - x^* \|^p \leq [1-\beta_n (p-1)] \| x_n - x^* \|^p + \beta_n \| T^n z_n - x^*) \|^p + [1-\beta_n (p-1)] + \beta_n\} N_5$$

$$\leq [1-\beta_n (p-1))] \| (x_n - x^*) \|^p + \beta_n k_n^p \| z_n - x^* \|^p + [1-\beta_n (p-2)] N_5 \quad (3.5)$$

Substituting (3.2) into (3.5) and setting $N_6 = \max [N_4, N_5]$, we have

$$\| y_n - x^* \|^p \leq [1-\beta_n (p-1)] \| x_n - x^* \|^p + \beta_n k_n^p [1-\gamma_n (p - k_n^p - 1)] \| x_n - x^* \|^p$$

$$+ \beta_n k_n^p [1 - \gamma_n (p-2)] N_6 + [1-\beta_n\}(p-2)] N_6$$

$$\leq \{[1-\beta_n (p-1) + \beta_n k_n^p [1 - \gamma_n (p - k_n^p - 1)]\} \| x_n - x^* \|^p$$

$$+ \beta_n k_n^p [1- \gamma_n(p-2)]N_6 + [1-\beta_n)(p-2)]N_6 \quad (3.6)$$

Finally,

$$\|x_{n+1} - x^*\|^p = \|a_n x_n + b_n T^n y_n + c_n u_n - x^*\|^p$$

$$= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n y_n - x^*) - c_n(T^n y_n - u_n)\|^p$$

$$\leq [1-\alpha_n(p-1)]\|(x_n - x^*) - c_n(T^n y_n - u_n)\|^p$$

$$+ \alpha_n c\|T^n y_n - x^*) - c_n(T^n y_n - u_n)\|^p - \alpha_n(1-\alpha_n^{(p-1)} c)\|T^n y_n - x^*) - (x_n - x^*)\|^p$$

Expanding further and observing that $\alpha_n(1-\alpha_n^{(p-1)} c) \geq 0$ for all $n > 0$, we have:

$$\|x_{n+1} - x^*\|^p \leq [1-\alpha_n(p-1)]\{\|x_n - x^*\|^p + cc_n\|T^n y_n - u_n)\|^p\}$$

$$+ \alpha_n c\{\|T^n y_n - x^*\|^p + cc_n\|T^n y_n - u_n)\|^p\}$$

$$= [1-\alpha_n(p-1)]\|x_n - x^*\|^p + \alpha_n c\|T^n y_n - x^*\|^p\}$$

$$+ [1-\alpha_n(p-1)c_n + \alpha_n cc_n]\|T^n y_n - u_n)\|^p \quad (3.7)$$

But T is asymtptically nonexpansive, $c_n < \alpha_n$ and $c < 1$. Therefore, simplifying (3.7), we have:

$$\|x_{n+1} - x^*\|^p \leq [1-\alpha_n(p-1)]\|x_n - x^*\|^p + \alpha_n k_n^p\|y_n - x^*\|^p$$

$$+ [1-\alpha_n(p-2)]\|T^n y_n - u_n\|^p \quad (3.8)$$

Continuity of T on the bounded set K implies that there exists a real number $N_7 < \infty$ such that $\|T^n y_n - u_n\|^p \leq N_7$. Substitute (3.5) into (3.8) and observe that $c_n < 1$, for all n, we have

$$\|x_{n+1} - x^*\|^p \leq [1-\alpha_n(p-1)]\|x_n - x^*\|^p + k_n^p \alpha_n\|y_n - x^*\|^p + [1-\alpha_n(p-2)]N_7$$

$$\leq [1-\alpha_n(p-1)]\|x_{n+1} - x^*\|^p + k_n^p \alpha_n\{[1-\beta_n(p-1)]$$

$$+ \beta_n k_n^p [1-\gamma_n(p - k_n^p - 1)]\}\|x_{n+1} - x^*\|^p + \alpha_n \beta_n \beta_n k^2{}_n^p [1-\gamma_n(p-2)]N_6$$

$$+ \alpha_n k_n^p [1-\beta_n(p-2)]N_6 + [1-\alpha_n(p-2)N_7$$

Let $N_8 = \max[N_6, N_7]$, then we have

$$\|x_{n+1} - x^*\|^p \leq \{[1-\alpha_n(p-1)] + k_n^p \alpha_n[1-\beta_n(p-1)] + \alpha_n \beta_n k_n^{2p}[1-\gamma_n(p-k_n^p-1)]\}\|x_n - x^*\|^p$$

$$+ \{\alpha_n \beta_n k_n^{2p}[1-\gamma_n(p-2)] + \alpha_n k_n^p[1-\beta_n(p-2)] + [1-\alpha_n(p-2)]\}N_8 \quad (3.9)$$

Let $\rho_n = \|x_n - x^*\|^p$. Then (3.9) becomes

$$\rho_{n+1} \leq (1-t_n)\rho_n + \sigma_n \quad (3.10)$$

where $t_n = \alpha_n(p-1)] - k_n^p \alpha_n[1-\beta_n(p-1)] - \alpha_n \beta_n k_n^{2p}[1-\gamma_n(p - k_n^p - 1)]$

and $\sigma_n = \{\alpha_n \beta_n k_n^{2p}[1-\gamma_n(p-2)] + \alpha_n k_n^p[1-\beta_n(p-2)] + [1-\alpha_n(p-2)]\}N_8$

From our hypothesis, it is clear that :

$$t_n = \alpha_n(p-1) - k_n^p \alpha_n[1-\beta_n(p-1)] - \alpha_n \beta_n k_n^{2p}[1-\gamma_n(p-k_n^p-1)]$$

$$= \alpha_n(p-1) - \alpha_n k_n^p + \alpha_n \beta_n k_n^p[p-1) - \alpha_n \beta_n k_n^{2p} + \alpha_n \beta_n \gamma_n k_n^{2p}(p-1-k_n^p)$$

$$= \alpha_n[p-1-k_n^p] + \alpha_n \beta_n k_n^p[p-1-k_n^p] + \alpha_n \beta_n \gamma_n k_n^{2p}(p-1-k_n^p)$$

$= [p-1-k_n^p][\alpha_n + \alpha_n \beta_n k_n^p + \alpha_n \beta_n \gamma_n k_n^{2p}]$

$\leq [p-1-k_n^p][1 + \beta_n k_n^p + \beta_n \gamma_n k_n^{2p}]$

≤ 1

We observe from our hypothesis that $t_n \geq 0$. Thus $t_n \in [0,1]$. Also, $\sum t_n = \infty$ and $\sigma_n = o(t_n)$. Hence, by Lemma 2.3, $\lim_{n\to\infty} \rho_n = 0.$ This implies that $\{x_n\}$ converges strongly to x^*. The proof is complete.

Remark:- By following similar procedure as in the proof of the above Theorem, we can be shown that the iteration method given by

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + c_n u_n \\ y_n &= a_n' x_n + b_n' T^n z_n + c_n' v_n \\ z_n &= a_n'' x_n + b_n'' T^n x_n + c_n'' w_n \end{aligned}\right\} \quad n \geq 1$$

where $\{v_n\}$, $\{w_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n'\}$, $\{a_n''\}$, $\{b_n\}$, $\{b_n'\}$, $\{b_n''\}$, $\{b_n''\}$, $\{c_n\}$, $\{c_n'\}$, $\{c_n''\}$, are real sequences in [0,1] satisfying certain conditions, also converges strongly to the fixed point of asymptotically non-expansive operator T.

We now consider the modified iteration procedure given by (1.6) for fixed points of asymptotically nonexpansive operators in uniformly smooth Banach spaces. Our result is the following.

Theorem 3.2:- Let B be a uniformly smooth Banach space and K a nonempty closed bounded and convex subset of B. Suppose S,T are uniformly continuous and asymptotically nonexpansive selfmapping of K with real sequence $\{k_n\}$ satisfying $k_n \geq 1, \forall_n$, and $\lim_{n\to\infty} k_n = 1$. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by

$$\left.\begin{aligned} x_{n+1} &= a_n x_n + b_n T^n y_n + cnS^n x_n \\ y_n &= a_n' x_n + b_n' S^n z_n + c_n' v_n \\ z_n &= a_n'' x_n + b_n'' T^n x_n + c_n'' w_n \end{aligned}\right\} \quad n \geq 1$$

where $\{v_n\}$, $\{w_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n'\}$, $\{a_n''\}$, $\{b_n\}$, $\{b_n'\}$, $\{b_n''\}$, $\{b_n''\}$, $\{c_n\}$, $\{c_n'\}$, $\{c_n''\}$, are real sequences in [0,1] satisfying the following conditions

(i) $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$,

(ii) $\sum b_n = \infty$,

(iii) $\alpha_n := b_n + c_n \beta_n := a_n' + b_n' + c_n' \Upsilon_n := b_n'' + c_n''$,

$0 \leq (p-1-k_n^p) \leq 1, \quad 1+\beta_n k_n^p (1+ \gamma k_n^p] \leq \dfrac{1}{p-1-k_n^p}.$

If S and T have a common fixed point in K, then the sequence $\{x_n\}$ converges strongly to the common fixed point of S and T.

Proof: By Goebel and Kirk [2], S and T have fixed points in K. Let $x_o \in K$ be the common fixed point of S and T. From our hypothesis and Lemma 2.2, we have the following estimates.

$$z_n - x_o \;\; {}^{p} = \;\; (1-\gamma_n)(x_n - x_o) + \gamma_n (T^n x_n - x_o) - c^{''}_n (T^n x_n - w_n) \;\; {}^{p}$$

$$\leq [1 - \gamma_n (p-1)] \;\; (x_n - x_o)) - c^{''}_n (T^n x_n - w_n) \;\; {}^{p}$$

$$+ \gamma_n c \;\; (T^n x_n - x_o) - c^{''}_n (T^n x_n - w_n) \;\; {}^{p}$$

$$- \gamma_n (1 - \gamma_n^{p-1} c) \;\; (T^n x_n - x_o) - (x_n - x_o) \;\; {}^{p}$$

Observe that $\gamma_n(1 - \gamma_n^{p-1} c) \geq 0$ and expand further, we have

$$z_n - x_o \;\; {}^{p} \leq [1-\gamma_n (p-1)]\{ \;\; x_n - x_o \;\; {}^{p} + cc^{''}_n \;\; T^n x_n - w_n \;\; {}^{p}$$

$$- \;\; p < cc^{''}_n (T^n x_n - w_n), j(x_n - x_o) > \}$$

$$+ \gamma_n c\{ \;\; T^n x_n - x_o \;\; {}^{p} - cc^{''}_n \;\; T^n x_n - w_n \;\; {}^{p}$$

$$- p < c^{''}_n (T^n x_n - w_n), j(x_n - x_o) > \}$$

$$\leq [1 - \gamma_n (p-1)]\{ \;\; x_n - x_o \;\; {}^{p} + cc^{''}_n \;\; T^n x_n - w_n \;\; {}^{p} \}$$

$$+ \gamma_n c\{ \;\; T^n x_n - x_o \;\; {}^{p} + cc^{''}_n \;\; T^n x_n - w_n \;\; {}^{p}\} \qquad (3.11)$$

By continuity of T and boundedness of K, there exists real number $N_9 < \infty$ such that $T^n x_n - w_n) \;\; {}^{p} \leq N_9$. Observe that $c, \; c^{''}_n < 1$. Then, from (3.11) we obtain

$$z_n - x_o \;\; {}^{p} \leq [1 - \gamma_n (p-1)] \;\; x_n - x_o \;\; {}^{p} + \gamma_n \;\; T^n x_n - x_o \;\; {}^{p} + [1 - \gamma_n (p-2)]N_9$$

$$\leq [1-\gamma_n (p - k_n^p - 1)] \;\; x_n - x_o \;\; {}^{p} + [1 - \gamma_n(p-2)]N_9 \qquad (3.12)$$

(since T is asymptotically nonexpansive) . Furthermore, we have

$$y_n - x_o \;\; {}^{p} \leq \;\; (1 - \beta_n)(x_n - x_o) + \beta_n (S^n z_n - x_o) - c^{'}_n (S^n z_n - v_n) \;\; {}^{p}$$

$$\leq [1 - \beta_n (p-1)] \;\; (x_n - x_o) - c^{'}_n (S^n z_n - v_n) \;\; {}^{p}$$

$$+ \beta_n c \;\; (S^n z_n - x_o) - c^{'}_n (S^n z_n - v_n) \;\; {}^{p}$$

$$- \beta_n (1 - \beta_n^{p-1} c) \;\; (S^n z_n - x_o) - (x_n - x_o) \;\; {}^{p} \qquad (3.13)$$

Expanding further and considering the fact that: $c, \; c^{''}_n \leq 1$ and $\beta_n(1 - \beta_n^{p-1} c) \geq 0$, we have

$$y_n - x_o \;\; {}^{p} \leq [(1 - \beta_n)(p-1)]\{ \;\; x_n - x_o \;\; {}^{p} + \;\; S^n z_n - v_n \;\; {}^{p}$$

$$- \;\; p < c_n (S^n z_n - v_n), j(x_n - x_o) > \}$$

$$+ \beta_n\{ \;\; S^n z_n - x_o \;\; {}^{p} + \;\; S^n z_n - v_n \;\; {}^{p}$$

$$- \;\; p < c_n (S^n z_n - v_n), j(S^n z_n - x_o) > \},$$

$$\leq [1 - \beta_n (p-1)]\{ \;\; x_n - x_o \;\; {}^{p} + \;\; S^n z_n - v_n \;\; {}^{p} \}$$

$$+ \;\; \beta_n\{ \;\; S^n z_n - x_o \;\; {}^{p} + \;\; S^n z_n - v_n \;\; {}^{p}$$

Since S is continuous on bounded set K, there exists a real number $N_o < \infty$ such that $S^n z_n - v_n \;\; {}^{p} \leq N_o$. Observing further that s is asymptotically nonexpansive, we obtain

$$y_n - x_o \;\; {}^{p} \leq [1 - \beta_n (p-1)] \;\; x_n - x_o \;\; {}^{p} + \beta_n \;\; S^n z_n - x_o \;\; {}^{p} + [1 - \beta_n (p-2)]N_o$$

$$\leq [1-\beta_n(p-1)] \; x_n - x^* \;^p + \beta_n k_n^p \; z_n - x_o \;^p + [1-\beta_n(p-2)]N_o \tag{3.14}$$

Let $Q_1 = \max[N_9, N_o]$ and substitute (3.12) into (3.14), we have

$$y_n - x_o \;^p \leq [1-\beta_n(p-1)] \; x_n - x_o \;^p + \beta_n k_n^p [1-\gamma_n(p-k_n^p-1)] \; x_n - x_o \;^p$$

$$+ \beta_n k_n^p [1-\gamma_n(p-2)]Q_1 + [1-\beta_n(p-2)]Q_1$$

$$\leq \{[1-\beta_n(p-1)] + \beta_n k_n^p [1-\gamma_n(p-k_n^p-1)]\} \; x_n - x_o \;^p$$

$$+ \beta_n k_n^p [1-\gamma_n(p-2)] + [1-\beta_n(p-2)]Q_1$$

$$= [1-\beta_n(p-1-k_n^p) - \beta_n \gamma_n k_n^p (p-k_n^p-1)] \; x_n - x_o \;^p$$

$$+ [1-\beta_n(p-2) + \beta_n k_n^p (1-\gamma_n(p-2))]Q_1 \tag{3.15}$$

Finally, we have the following estimates.

$$x_{n+1} - x_o \;^p = \; a_n x_n + b_n T^n y_n + c_n S^n x_n - x_o \;^p$$

$$= \; (1-\alpha_n)(x_n - x_o) + \alpha_n(T^n y_n - x_o) - c_n(T^n y_n - S^n x_n) \;^p$$

$$\leq [1-\alpha_n(p-1)] \; (x_n - x_o) - \alpha_n(T^n y_n - S^n x_n) \;^p$$

$$+ \alpha_n c \; (T^n y_n - x_o) - c_n(T^n y_n - S^n x_n) \;^p$$

$$- \alpha_{n\,(1-}\alpha_n{}_n^{p-1} c) \; (T^n y_n - x_o) - (x_n - x_o) \;^p$$

But $\alpha_n(1 - \alpha_n{}_n^{p-1} c) \geq 0$, therefore

$$x_{n+1} - x_o \;^p \leq [1-\alpha_n(p-1)]\{ \; (x_n - x_o) \;^p + cc_n \; T^n y_n - S^n x_n \;^p$$

$$- p < c_n(T^n y_n - S^n x_n), \; j(x_n - x_o) > \}$$

$$+ \alpha_n c\{ \; (T^n y_n - x_o) \;^p + cc_n \; T^n y_n - S^n x_n \;^p$$

$$- p < c_n(T^n y_n - S^n x_n), \; j(T^n y_n - x_o) > \}$$

$$\leq [1-\alpha_n(p-1)] \; x_n - x_o \;^p + [1-\alpha_n(p-2)] \; T^n y_n - S^n x_n \;^p + \alpha_n k_n^p \; (y_n - x_o \;^p \tag{3.16}$$

T and S are completely continuous on K, therefore there exists a real number $Q_2 < \infty$ such that $T^n y_n - S^n x_n \;^p \leq Q_2$. Substituting (3.15) into (3.16), we have

$$x_{n+1} - x_o \;^p \leq [1-\alpha_n(p-1)]\{ \; x_n - x_o \;^p + [1-\alpha_n(p-2)]Q_2$$

$$+ \alpha_n k_n^p [1-\beta_n(p-1-k_n^p)(1+\gamma_n k_n^p] \; x_n - x_o \;^p$$

$$+ \alpha_n k_n^p [1-\beta_n(p-2-k_n^p) + \beta_n k_n^p \gamma_n]Q_1$$

$$= \{[1-\alpha_n(p-1)] + \alpha_n k_n^p [1-\beta_n(p-1-k_n^p)(1+\gamma_n k_n^p)]\} \; x_n - x_o \;^p + \sigma_n \tag{3.17}$$

where $\sigma_n = [1-\alpha_n(p-2)]Q_2 + \alpha_n k_n^p [1-\beta_n(p-2-k_n^p) \; 1 + \beta_n k_n^p \gamma_n(p-2)]Q_1$

Let $t_n = \alpha_n(p-1-k_n^p)[1-\beta_n k_n^p)(1+\gamma_n k_n^p)]$

From our hypothesis, we see that $t_n \in [0,1]$. Also $\sum t_n = \infty$ and $\sigma_n = o(t_n)$.
Now, let $x_n - x_o \;^p = \Phi_n$, then (3.17) reduces to

$$\Phi_{n+1} \leq (1-t_n)\Phi_n + \sigma_n$$

Hence, by Lemma 2.3, $\Phi_n \to 0$ as $n \to \infty$. This implies that $\{x_n\}$ converges strongly to x_o – the common fixed point of S and T. The proof is complete.

Remark: Theorem 3.1 is an improvement and generalization of the results of Schu [10], Osilike and Igbokwe [6], as well as Xu and Noor [13] in the sense that the modified Mann and Ishikawa iteration methods used by these authors are special cases of our modified three-step iteration methods. Indeed, if in (1.7), $b''_n = c''_n = 0$, then it will reduce to the modified Ishikawa iteration method (1.4) which itself contains (1.3) as a special case. Also if $c_n = c'_n = c''_n = 0$, then (1.7) will reduce to Xu and Noor's iteration method (1.5) with $\alpha_n = b_n$, $\beta_n = b'_n$ and $\gamma_n = b''_n$. We further observe that if

$c_n = c'_n = c''_n = b''_n = 0$, in (1.7), then it will reduce to the modified Ishikawa

Iteration method without errors. Furthermore, Theorem 3.2 is a generalization Theorem 3.1 since the iteration method (1.7) is a special case of (1.6) and Theorem 3.2 investigates the common fixed point of the two operators involved when they are both asymptotically nonexpansive. Hence our results are improvements and generalizations of the results of Schu[10], Osilike and Igbokwe [6], Xu and Noor[13], as well as Owojori and Imoru[7] and other relevant results to the more general modified three-step iteration methods.

CONCLUSION

In this work, we have been able to establish strong convergence results for some improved modified three-step iteration methods for fixed points and common fixed points of asymptotically nonexpansive mappings in uniformly smooth Banach spaces. Our iteration methods here are generalizations of the previous modified Mann and Ishikawa iteration methods introduced by Schu[10] and Xu and Noor[13].

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