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# A Discrete Optimization Description for the Solutions in the Matching Problem

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**Abstract**: This study was concerned with the characterization of solutions in the matching problem. The general mixed-integer programming problem is given together with the definition of the convex hull of the integer solutions. In addition, the matching problem is defined as an integer problem and an algorithm is described to find the optimum matchings. Some illustrative examples are introduced to clarify the presented theory in the study.

**Key words**: Mixed-integer programming, convex hull, matching problem

## INTRODUCTION

Shortly after the development of his simplex algorithm for linear programs, Dantzig[1] pointed out the significance of solving integer programming problems. Many problems involving nonconvex region or functions can be converted into integer programs, e.g. linear programs with separable, piecewise linear, but nonconvex objective functions. The challenge raised by Dantzig was to develop effective procedures, such as his simplex method was proving to be for linear programs, to handle these more difficult integer programs.

Today, there are codes, including commercial codes, which effectively solve a small, but important, class of integer programs. This class is not well defined but can be said to include integer programs with linear programming relaxations, i.e. up to several thousand rows and columns, but with a small number of integer variables $x_j$, $j \in J$, or a strong linear programming relaxation.

**Mixed-integer programming problem:** The general mixed-integer programming problem can be formulated mathematically as follows[2]:

Maximize $z = cx$,

Subject to

$Ax = b$, (1)

$x_j \geq 0$, $(j = 1, 2, \ldots, n)$ (2)

$x_j$ integer, $j \in J \subseteq \{1, 2, \ldots, n\}$, $J \neq \phi$. (3)

An integer programming problem is a pure integer problem if $J = \{1, 2, \ldots, n\}$. The linear programming relaxation of the integer programming problem is the corresponding linear program with constraints (1) and (2) imposed, but not (3).

**Strong linear programming relaxations:** The considerations given here turn out to be very important practically. Any method that solves linear programs as part of a method to try to solve integer programs will profit from a better linear programming formulation.

Earlier, we defined the integer programming problem and its linear programming relaxation. Correspondingly, define the convex hull of integer solutions to be

$P_I = \text{conv}\{x \mid x_j \geq 0, j = 1, 2, \ldots, n; x_j \text{ integer}, j \in J; Ax = b\}$ (4)

and the linear programming polyhedron to be

$P_L = \{x \mid x_j \geq 0, j = 1, 2, \ldots, n; \text{ and } Ax = b\}$ (5)

Clearly, $P_I \subseteq P_L$.

There are many linear programs giving the same $P_I$ but different $P_L$'s and this is illustrated in the following examples:

**Example 1:** Consider

$P_I = \text{conv}\{(x_1, x_2) \mid 2x_1 \leq 7, 4x_2 \leq 9, x_1 \geq 0, x_2 \geq 0 \text{ and integers}\}$,

and

$P_{L_1} = \{(x_1, x_2) \mid 2x_1 \leq 7, 4x_2 \leq 9, x_1 \geq 0, x_2 \geq 0\}$,

$P_{L_2} = \{(x_1, x_2) \mid x_1 \leq 3, x_2 \leq 2, x_1 \geq 0, x_2 \geq 0\}$.

The polyhedra $P_I$, $P_{L_1}$ and $P_{L_2}$ are shown in Fig. 1a and 1b.

It is clear that $P_I = P_{L_2}$, while $P_I \subset P_{L_1}$.

**Example 2**: Consider the constraints:

$x_1 + x_2 \leq 2y$, $0 \leq x_j \leq 1$, $j = 1, 2$ and $y = 0$ or $1$.

The two polyhedra $P_I$ and $P_L$ are shown in Fig. 2a and 2b.

$P_I = \{(x_1, x_2, y) \mid 0 \leq x_1 \leq y, 0 \leq x_2 \leq y, 0 \leq y \leq 1\}$.

If we had originally stated the problem as having constraints

$0 \leq x_1 \leq y$, $0 \leq x_2 \leq y$, $y = 0$ or $1$,

then the linear programming relaxation would have had $P_L = P_I$.

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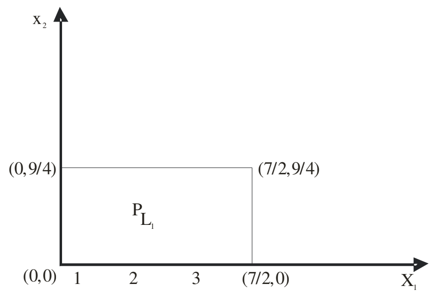



Fig. 1a

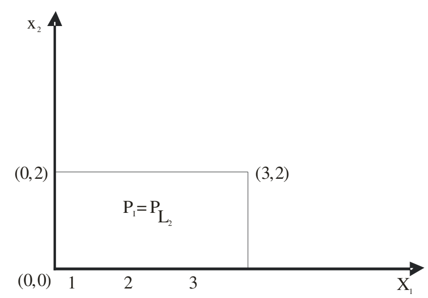



Fig. 1b

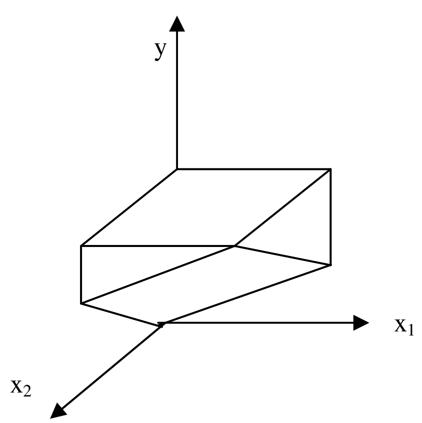



Fig. 2a

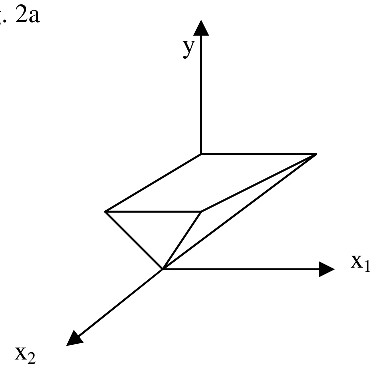



Fig. 2b

**Combinatorial Polyhedra:** Now, we defined $P_I$ to be the convex hull of integer solutions ($x_j$ integer for $j \in J$) of a linear program. Here, we consider $P_I$ to be the convex hull of some combinatorially defined polyhedra.

The prototype of this study is Edmonds[3,4] work on the matching polytope. His work went beyond the network flow class and yet converted a combinatorial optimization problem into a linear program. Even though that linear program has an enormous number of inequalities, he described a good (polynomially bounded) algorithm for solving the matching problem.

In what follows, the matching problem formulation is given and the solution concept is defined.

**The matching problem formulation:** Given an undirected graph with vertices V and edges E, the matching problem is to find subset M of edges so that no two edges of M meet the same vertex and maximize $\sum_{e \in M} c(e)$ over all such M, where ($c(e)$, $e \in E$) is given as the objective function.

The matching problem can be formulated mathematically as the following integer program

$$\text{Maximize } \sum_{e \in E} c(e)\, x(e) \qquad (6a)$$

Subject to

$$\sum_{e \text{ meets } v} x(e) \le 1\,, \qquad (6b)$$

$$x(e) \ge 0 \text{ and integer, } e \in E. \qquad (6c)$$

However, the matching polytope $P_I$ is

$P_I = \text{conv}\{x(e), e \in E \mid x(e) = 0 \text{ or } 1 \text{ and } M = \{e \mid x(e) = 1\} \text{ is matching}\}$.

There are various linear programming relaxations of the matching problem, but the focus here is to begin with $P_I$ and try to learn as much as possible about it. In particular, we would like to characterize all of its facets. This characterization was given by Edmonds as

$$\sum_{e \in D(S)} x(e) \le \frac{|S|-1}{2}\,, \qquad (7)$$

where S is a subset of nodes of odd cardinality and D(S) is the subset of edges $e \in E$ with both ends of e meeting nodes of S. That is, $P_I$ is equal to the solution set

$$\{x(e),\ e \in E \mid\ x(e) \ge 0,\ \sum_{e \text{ meets } v} x(e) \le 1 \text{ and } \sum_{e \in D(S)} x(e) \le \frac{|S|-1}{2}\}.$$

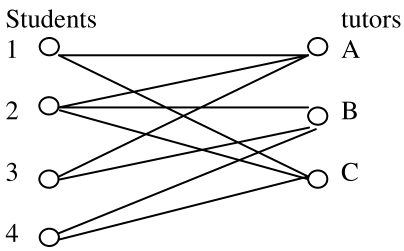
When we refer to the matching polytope, we mean the convex hull of incidence vectors of sets M which are matchings. When we say the linear characterization of the matching polytope, we mean the description of all additional inequalities to convert the problem to a linear program as Edmonds did for the matching problem.

However, Edmonds' algorithm for finding optimum matchings did not depend on knowing which were actually facets. In any case, the facets were among the set given.

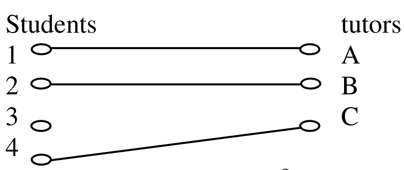
Other polytopes whose linear characterizations have been found are the polytope of subsets M of E which are independent in two matroids $M_1$ and $M_2$ over E[5] and the polytope of collections of edges which are of even degree at certain nodes and of odd degree at the other nodes[6].

**Example 3:** At the end of an academic year, each student has to take an examination with each of his or her teacher. How many examination period are required?

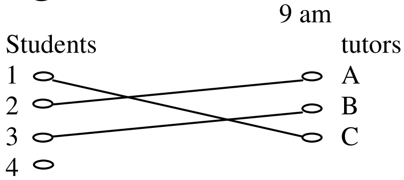
We can see what is involved if we consider a simple example with four students and three teachers. We represent the students and teachers by the vertices of a bipartite graph and join a student-vertex to a teacher-vertex whenever the student needs to be examined by the teacher. An example of such a graph is:




If two edges meet at a common vertex, then the corresponding examination cannot take place simultaneously. So the problem reduces to that of splitting the graph into subgraphs in which no two edges meet in a common vertex-that is, into matchings. In this particular case, the minimum number of matchings which decompose the graph is 3 and a suitable timetable is as follows:




9 am

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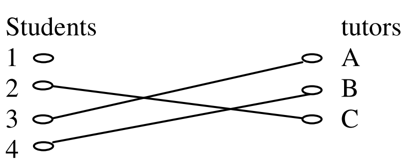


11 am

The corresponding edge decomposition is
{1A, 2B, 4C}, {1C, 2A, 3B}, {2C, 3A, 4B}.

Note that this can also be thought of as an edge-coloring problem. If we color the 9 *am* edges red, the 10 *am* edges yellow and the 11 *am* edge bleu, then the colors appearing at each vertex (student or teacher) are different.

In these scheduling problems the graphs under consideration are all bipartite graphs. The problem therefore reduces to that of finding the chromatic index of a bipartite graph and this problem is answered completely by König's theorem (theorem 12.8)[7] the smallest number of matchings needed is equal to the largest vertex-degree in the bipartite graph. Thus the matching problem is solved in this case.

## CONCLUSION

It has been shown that the matching problem can be converted into a pure integer programming problem. The characterization of solutions for such problem has been given. Some illustrative examples have been introduced to clarify the developed theory in the study. However, there are many other aspects should be studied in the area. Some of these points are:

A computer code is needed to test how the procedure works for the characterization of solutions in the matching problem.

A study is required to deal with the matching problem under randomness and fuzziness.

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