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# Behavior of the Dedekind's Function over First Order Theta Function According to Conditions Modular Form

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**Abstract:** The effect Dedekind's etha function on theta functions is analyzed according to the characteristics of theta functions under modular group conditions.

**Key words:** Theta functions, characteristic values, modular group, dedekind functions

## INTRODUCTION

By $SL_2$ we mean the group of 2x2 matrices with determinant 1. We write $SL_2(R)$ for those elements of $SL_2$ having coefficients in a ring R. In practice, the ring R will be integers Z, rational numbers Q and real numbers R. We call $SL_2(Z)$ the modular group $\Gamma$.

If L is lattice in complex numbers-C, then we can always select a basis, L = ( $\omega_1, \omega_2$ ) such that $\tau = \frac{\omega_2}{\omega_2}$ is an element of the upper half-plane $\aleph$, i.e. has $im\tau \succ 0$ which is not real. If D consist of all $u \in \aleph$ such that $-\frac{1}{2} \le \operatorname{Re} u \le \frac{1}{2}$, $|u| \ge 1$ and T=$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \Gamma$, S=$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$ then D is a fundamental domain for modular group $\Gamma$ in $\aleph$, Then, S,T $\in \Gamma$ generate modular group $\Gamma$[1]. We define characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ where $\varepsilon, \varepsilon'$ are integers according to characteristics $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\bmod 2)$ but $\theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,\tau) \equiv 0$ for theta function $\theta$. If n is any positive integer we define $\Gamma_0(n)$ to be the set of all matrices A = $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in modular group $\Gamma$ with $\gamma \equiv (\bmod n)$ and but

$$\Gamma_0(n) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1) : \gamma \equiv (\bmod n) \right\}$$ [2,3]

It is easy to verify that $\Gamma_0(n)$ is a subgroup $\Gamma$.
If We consider the congruence subgroup $\Gamma(2)$, Then

$$\Gamma_X(2) = \{ W \in \Gamma(1) : \quad W \equiv I \quad , \quad W \equiv X (\bmod 2) \ \}$$

where I is the unit matrix and for matrices $X = S$, $X = T$, $X = U$ the three subgroups $\Gamma_S(2)$, $\Gamma_T(2)$, $\Gamma_U(2)$ are conjugate subgroups of $\Gamma(1)$ for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

We shall need to study such groups when we introduce theta functions.

We note that the above matrices, defined V = ST = $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $V^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ = I , U = TST form a set coset representatives of $\Gamma(1)$ modulo $\Gamma(2)$[4].

The subgroup $\wp(n)$ of $\Gamma(1)$ is generated by $V^2$ and S where k is an odd positive integer and the set of elements in $\wp(k)$ of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

**Theorem 1:** Let n be any prime and $S\tau = -\frac{1}{\tau}$, $T\tau = \tau + 1$ be the generations of the full modular group $\Gamma$, then for every $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $W \notin \Gamma_0(n)$ there exists an element $K = \begin{pmatrix} p & r \\ s & t \end{pmatrix} \in \Gamma_0(n)$.

**Proof:** If $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ where $c$ is not $c \equiv 0 (\bmod n)$ then we wish to find $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, with $s \equiv 0 (\bmod n)$ and an integer q , $0 \le q \le n$, such that

$$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p & r \\ s & t \end{pmatrix} ST^q$$

$$= \begin{pmatrix} p & r \\ s & t \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & r \\ s & t \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & q \end{pmatrix}.$$

All matrices here are nonsingular so we can solve for $\begin{pmatrix} p & r \\ s & t \end{pmatrix}$ to get

$$\begin{pmatrix} p & r \\ s & t \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & w \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} qa-b & a \\ qc-d & c \end{pmatrix}.$$

Choose q to be that solution of the congruence $qc \equiv d (\bmod n)$ with $0 \le q \le n$. This is possible since $c$ is not $c \equiv 0(\bmod n)$, now take s = qs-t , p = wp-r , r = a , t = c , then $s \equiv 0(\bmod n)$ so $K = \begin{pmatrix} p & r \\ s & t \end{pmatrix} \in \Gamma_0(n)$.

We define the first order theta function with characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$, $u \in C$ and theta period $\tau$ by

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u,\tau)$$
$$= \sum_{N=-\infty}^{\infty} \exp\left\{(N+\frac{\varepsilon}{2})^2 \pi i \tau + 2\pi i (N+\frac{\varepsilon}{2})(u+\frac{\varepsilon'}{2})\right\}$$

where N is a integer[5].

It has been seen at several points that the theta functions whose characteristics are pair of integers $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ satisfy simpler identities than those for which $\varepsilon, \varepsilon'$ are general real numbers. As $\varepsilon, \varepsilon'$ are residue classes (mod2) it is natural to concentrate attention on the four functions $\theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(u,q)$, $\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(u,q)$, $\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(u,q)$ and $\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u,q)$ which we shall call the four principal theta functions. For any integers m,n, when $\varepsilon, \varepsilon'$ are integers, we have

$$\theta\begin{bmatrix} \varepsilon + 2m \\ \varepsilon' + 2n \end{bmatrix}(u,q) = (-1)^{n\varepsilon}\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u,q).$$

When $\varepsilon, \varepsilon'$ are integers , the theta series defined by

$$\theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(u,q) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{2i(n+\frac{1}{2})(u-\frac{\varepsilon'\pi}{2})}$$

can be converted into fourier series by pairing off the terms which $n+\frac{\varepsilon}{2}$ has equal and opposite values, n with -n if $\varepsilon = 0$ and n with -(n+1), leaving in the former case an unpaired term for n= 0, whose values is 1. The terms so paired have a common factor $q^{(n+\frac{\varepsilon}{2})^2}$ and the sum of their remaining factors is $2Cos(2n+\varepsilon)(u-\frac{\varepsilon'\pi}{2})$. Thus we have the four series

$$\theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(u,q) = 2\sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} Sin(2n+1)u$$

$$\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(u,q) = 2\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} Cos(2n+1)u$$

$$\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(u,q) = 1 + 2\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} Cos2nu$$

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u,q) = 1 + 2\sum_{n=-\infty}^{\infty} q^{n^2} Cos2nu.$$

Moreover,

$$\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(u) = \theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(u+\frac{\pi}{2})$$

$$\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(u) = -iq^{\frac{1}{4}}e^{iu}\theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(u+\frac{\pi\tau}{2})$$

$$\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u) = q^{\frac{1}{4}}e^{iu}\theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(u+\frac{\pi+\pi\tau}{2}).$$

If N is a positive integer then theta function order n or n[th] is defined by

$$\theta^n\begin{bmatrix} \mu \\ \mu' \end{bmatrix}(u,\tau) = \sum_{M=-\infty}^{\infty} C_M \theta\begin{bmatrix} \dfrac{2(M+\frac{\mu}{2})}{N} \\ \mu \end{bmatrix}(Nu, N\tau)$$

where $0 \le M \le N-1$.

In fact, An theta function order n my be found by taking the product of n first theta functions. Its characteristic $\begin{bmatrix} \mu \\ \mu' \end{bmatrix}$ is given by the matrix sum of the n characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$. The $C_M$ is indepent on u put my depent on $\tau$.

$C_M$ satisfy $C_{NK+M} = C_M .\exp(\pi i) . \Phi(K)$

where

$$\Phi(K) = N\tau\left(K + \frac{(M+\frac{\varepsilon}{2})}{N}\right)^2 + N\left(K + \frac{(M+\frac{\varepsilon}{2})}{N}\right)\frac{\varepsilon'}{N}.$$

Functions $\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(u,\tau)$ has zeros at the points $u = \frac{1}{2} - \frac{1}{2}\tau + r_1 + r_2\tau$. These points form a lattice, that 1-exp$\left[(2k-1)\tau + 2u\right]$ has zeros at points u where $(2k-1)\tau \equiv 1(\bmod 2)$ or equivalently. Hence function theta order n defined by

$$\Phi(u,\tau) = \prod_{1}^{\infty}\left\{1 + \exp \pi i\left[(2k-1)\tau - 2u\right]\right\}$$

$$\prod_{1}^{\infty}\left\{1 + \exp \pi i\left[(2k-1)\tau + 2u\right]\right\}$$

has precisely the same zeros as first order theta $\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)$ provided the product converges. Thus we have absolute and uniform convergence of the first infinite product for $im\tau \succ 0$. For periods 1 and $\tau$, we have

$$\Phi(u+1,\tau)=\prod_1^\infty\left\{1+\exp\pi i\left[(2k-1)\tau+2u+2\right]\right\}$$

$$\prod_1^\infty\left\{1+\exp\pi i\left[(2k-1)\tau-2u-2\right]\right\}=\Phi(u,\tau)$$

$$\Phi(u+\tau,\tau)=\prod_1^\infty\left\{1+\exp\pi i\left[(2k-1)\tau+2u\right]\right\}$$

$$\prod_1^\infty\left\{1+\exp\pi i\left[(2k-1)\tau-2u\right]\right\}$$

Setting $q=\exp\pi i\tau$ we may write

$$\Phi(u,\tau)=\prod_1^\infty\left[1+q^{2k-1}\exp 2\pi iu\right]$$

$$\prod_1^\infty\left[1+q^{2k-1}\exp(-2\pi iu)\right]$$

It was introduced by Dedekind function $\eta(\tau)$ and is defined by the equation

$$\eta((\tau)=e^{\frac{\pi i\tau}{12}}\prod_{n=1}^\infty\left(1-e^{2\pi in\tau}\right).$$

The Dedekind function $\eta(\tau)$ is cups form of weight $\frac{1}{2}$ on $\Gamma(1)$ and satisfy

$$\eta(A\tau)=v_\eta(A)(\gamma\tau+\delta)^{\frac{1}{2}}\eta(\tau)$$

for all A = $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma(1)$ [6].

Dedekind proved the following law oh transformation of $\log\eta(\tau)$ under the action of the elliptic modular group. If A = $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma$ then we have $\log\eta\left(A(\tau)\right)=\log\eta(\tau)+\frac{1}{12}\pi i\beta \quad for \quad \gamma=0$

and

$$\log\eta\left(A(\tau)\right)=\log\eta(\tau)+\frac{1}{2}\log\left(\frac{\gamma\tau+\delta}{i}\right)$$

$$\frac{1}{12\gamma}\pi i(\alpha+\delta)-\pi ig(\gamma,\delta), \quad for \quad c>0$$

where $A(\tau)=\frac{\alpha\tau+\beta}{\gamma\tau+\delta}$, all logarithms are taken with respect to the principal branch and $g(\gamma,\delta)$ is a Dedekind sum[7].

An important connection between $\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)$ and $\eta(\tau)$ is given by

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\equiv\eta^2(\frac{\tau+1}{2})/\eta(\tau+1).$$

The infinite product has the form $\prod(1-u^n)$ where $u=e^{2\pi i\tau}$. If $\tau\in\aleph$ then $|u|\prec 1$ so the product converges absolutely and non-zero.

Moreover, since the convergence is uniform on compact subsets of $\aleph$, $\eta(\tau)$ is analytic on $\aleph$. This result and other properties of $\eta(\tau)$ following from transformation formulas which describe the behavior of $\eta(\tau)$ under elements of the modular group $\Gamma$.

i. For the generator $T\tau=\tau+1$ we have

$$\eta(\tau+1)=e^{\frac{\pi i(\tau+1)}{12}}\prod_{n=1}^\infty\left(1-e^{2\pi in(\tau+1)}\right)$$

ii. For the other generator $S\tau=-\frac{1}{\tau}$ we have the

$$\eta(-\frac{1}{\tau})=\left(-i\tau\right)^{\frac{1}{2}}.\eta(\tau).$$

For proof, let $\tau=iy$ where y >0 and then extend the results to all $\tau\in\aleph$ by analytic continuation. The transformation formula becomes $\eta(i/y)=y^{\frac{1}{2}}\eta(iy) \quad for \quad \tau=iy$ and this is equivalent to $\log(i/y)-\log\eta(iy)=\frac{1}{2}\log y$.

and

$$\log\eta(iy)=-\frac{1}{12}\pi y+\log\prod_{n=1}^\infty(1-e^{-2\pi ny})$$

$$=-\frac{1}{12}\pi y+\sum_{n=1}^\infty(1-e^{-2\pi ny})=-\frac{1}{12}\pi y-\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{e^{-2\pi mny}}{m}$$

$$=-\frac{1}{12}\pi y-\sum_{m=1}^\infty\frac{1}{m}\left(\frac{e^{-2\pi my}}{1-e^{2\pi my}}\right)$$

we obtained $\eta(-\frac{1}{\tau})=\left(-i\tau\right)^{\frac{1}{2}}\eta(\tau)$ since

$$\sum_{m=1}^\infty\frac{1}{m}\left(\frac{1}{1-e^{2\pi my}}\right)-\sum_{m=1}^\infty\frac{1}{m}\left(\frac{1}{1-e^{2\pi m/y}}\right)_{[2]}$$

$$-\frac{1}{12}\pi(y-\frac{1}{y})=-\frac{1}{2}\log y$$

**Lemma 1:** Let $\Gamma$ be a subgroup of $\Gamma(1)$. If $\varphi(\tau)$ is a modular form of weight for $\Gamma$ with multiplier system $t$ then we write $\varphi(\tau)\in A(\Gamma,n,t)$. If $\varphi(\tau)\in A(\Gamma,n,t)$ then the

$\psi$-transform $\varphi_\psi$ of $\varphi$ is defined by

$$\varphi_\psi(\tau)=\varphi(\tau)/\psi=\left\{\xi(\psi,\tau)\right\}^{-1}\varphi(\psi,\tau)$$

Here, $\xi(\psi,\tau)=(\gamma\tau+\delta)^n$ where $A=\begin{pmatrix} * & * \\ \gamma & \delta \end{pmatrix}$

If $\varphi_1(\tau)\in A(\Gamma,n_1,t_1)$ *and* $\varphi_2(\tau)\in A(\Gamma,n_2,t_2)$ then we have

$\varphi_1(\tau).\varphi_2(\tau)\in A(\Gamma,n_1+n_2,t_1.t_2)$

$\frac{\varphi_1(\tau)}{\varphi_2(\tau)}\in A(\Gamma,n_1-n_2,\frac{t_1}{t_2})$ .

Let $k$ be a prime number greater than 3. If $\sigma$ is a even integer such that $\sigma(k-1)\equiv 0(Mod\,24)$ , then $\Theta(\tau)=\left[\frac{\eta(k,\tau)}{\eta(\tau)}\right]^\rho$ is a modular function on the group $\Gamma_0(k)$. The multiplier system $\rho$ of $\Theta(\tau)$ is given by $\rho(A)=\left[\frac{\delta}{k}\right]^\rho$ where $A=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\in\Gamma_0(k)$ and $\left[\frac{\delta}{k}\right]$ is Legendre's symbol.

**Lemma 2:** The functions $\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)$ , $\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)$ and $\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)$ are entire modular form of weight $\frac{1}{2}$ for the groups $\Gamma_S(2)$, $\Gamma_T(2)$ and $\Gamma_U(2)$, respectively. Further,

$\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\mid K = e^{-\frac{\pi i}{4}}\theta\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)$

$\theta\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)\mid K^2 = e^{-\frac{\pi i}{2}}\theta\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)$

Also , for n $\geq$ 0

The functions $\theta^n\begin{bmatrix}0\\1\end{bmatrix}(0,\tau)$ , $\theta^n\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)$ and $\theta^n\begin{bmatrix}1\\0\end{bmatrix}(0,\tau)$ are entire modular form of weight $\frac{n}{2}$ for the groups $\Gamma_S(2)$, $\Gamma_T(2)$ and $\Gamma_U(2)$, respectively.

**Theorem 2:** Let $k$ be a prime number greater than 3 and $\sigma$ is a even integer such that $\sigma(k-1)\equiv 0(Mod\,24)$ and put r = $n\rho$ where n is a positive integer. If the characteristics $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ and $\begin{bmatrix}\mu\\ \mu'\end{bmatrix}$ are $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix},\begin{bmatrix}\mu\\ \mu'\end{bmatrix}\equiv\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}(\text{mod }2)$, the $\phi_{kr}(\tau)$ is a modular function on the group $\Gamma_0(k)$. The multiplier system $\rho$ of $\Theta(\tau)$ is given by $\rho(A)=\left[\frac{\delta}{k}\right]^\rho$ where $A=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\in\Gamma_0(k)$ and $\left[\frac{\delta}{k}\right]$ is Legendre's symbol.

**Proof:** $\phi_{kr}(\tau)\ \neq 0$ is regular in ℵ . If each positive integers $\sigma$ , $n$ and even positive integer r $=n\rho$. Therefore, the characteristics $r^{th}$ order theta functions are $\begin{bmatrix}r\varepsilon\\ r\varepsilon'\end{bmatrix},\begin{bmatrix}r\mu\\ r\mu'\end{bmatrix}\equiv\begin{bmatrix}0\\0\end{bmatrix}(\text{mod }2)$ , then we have

$$\phi_{kr}(\tau)=\frac{\theta^{n\rho}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0,k\tau)}{\theta^{n\rho}\begin{bmatrix}\mu\\ \mu'\end{bmatrix}(0,\tau)}=\frac{\theta^{n\rho}\begin{bmatrix}0\\0\end{bmatrix}(0,k\tau)}{\theta^{n\rho}\begin{bmatrix}0\\0\end{bmatrix}(0,\tau)}=$$

$$\left[\frac{\eta^2\left(\frac{k\tau+k}{2}\right)/\eta(k\tau+k)}{\eta^2\left(\frac{\tau+1}{2}\right)/\eta(\tau+1)}\right]^{n\rho}=\left[\frac{\eta^2\left(\frac{k\tau+k}{2}\right)/\eta^2(\frac{\tau+1}{2})}{\eta^2(k\rho)/\eta(\tau+1)}\right]^{n\rho}$$

Setting $\lambda=\frac{\tau+1}{2}$ and observing that $\Phi(\lambda)=\left[\frac{\eta(k\lambda)}{\eta(\lambda)}\right]^\rho$, we obtained $\phi_{kr}(\tau)=\left[\frac{\Phi^2(\lambda)}{\Phi^2(2\lambda)}\right]^n$

By Lemma1 and from the equation $\Phi(A\tau)=\left[\frac{\delta}{k}\right]^\rho.\Phi(\tau)$, we have

$$\phi_{kr}(A\tau)=\left[\frac{\Phi^2(A\lambda)}{\Phi^2(2\lambda)}\right]^\rho=\left[\frac{\left(\frac{\delta}{k}\right)^{2\rho}\Phi^2(\lambda)}{\left(\frac{\delta}{k}\right)^{\rho}\Phi(2\lambda)}\right]^n=\left(\frac{\delta}{k}\right)^\rho\phi_{kr}(\tau)\cdot$$

Finally, we consider the expansions of $\phi_{kr}(\tau)$ at the parabolic cusp $\infty$ and 0 . Hence, We have

$\Phi(\tau)=\exp\left[\frac{\pi i(k-1)\rho}{12}\right]\left[1+\sum_{N=1}^{\infty}C_N.e^{-2\pi iN\tau}\right]$ as the Fourier expansion of $\Phi(\lambda)$ function at $\infty$ . $\phi_{kr}(\tau)$ has the Fourier expansion at $\infty$ of the form

$$\phi(\tau)=1+\sum_{N=1}^{\infty}C_N.e^{2\pi iN\tau}$$

$$\Phi(\tau)=k^{-\frac{\rho}{2}}\exp\left[\frac{\pi i(k-1)\rho}{12k\tau}\right]\left[1+\sum_{N=1}^{\infty}H_N.e^{-2\pi iN/k\tau}\right]$$

as the Fourier expansion at 0. Hence

$$\phi_{kr}(\tau)=\exp\left[\frac{r\pi i(k-1)}{8k\tau}\right]\left[1+\sum_{N=1}^{\infty}R_N.e^{-2\pi iN/k\tau}\right].$$

It follows that $\phi_{kr}(\tau)$ is a modular function on $\Gamma_0(k)$.

**Theorem 3:** $\eta\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)=P(\alpha,\beta,\gamma,\delta)[-i(\gamma\tau+\delta]^{\frac{1}{2}}\eta(\tau)$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\in\Gamma$ , $\gamma$>0 and

$$P(\alpha,\beta,\gamma,\delta)=\exp\left\{\pi i\left[\frac{\alpha+\delta}{12\gamma}+q(-\delta,\gamma)\right]\right\}$$

and

$$q(h,k)=\sum_{r=1}^{k-1}\frac{r}{k}\left[\frac{hr}{k}-\left(\frac{hr}{k}\right)-\frac{1}{2}\right]$$

Note: The sum $q(h,k)$ is called a Dedekind sum .

**Theorem 4:** The set of modular forms, the entire modular forms and the cups forms each of same dimension for $\Gamma(1)$, form vector space over the complex field.

Let g be a homogeneous modular form of dimension -k for the group $\Gamma$ in the variables $\omega_1,\omega_2$. We write this in the form $g\left[\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right]$ and consider $\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$ as a matrix. We define the function $g_B$ by

$$g_B\left[\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right] = g\left[B\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right]$$

where $M=\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}$ , $\alpha>0$ , $\alpha\gamma=$ n and $im\frac{\omega_1}{\omega_2}>0$ and call it a transform of $g$ of order n. It satisfies the following equations

$$g_B\left[\lambda\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right]=\lambda^{-k}\quad g_B\left[\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right] \text{ for } \lambda\in C \quad , \quad \lambda\neq 0$$

$$g_B\left[M\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right]= g_B\left[\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right] \text{ for } M=\begin{pmatrix}\alpha & -\beta\\ -\lambda & \delta\end{pmatrix}\in\Gamma_B$$

$$g_{MB}\left[M\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right]= g_B\left[\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right] \text{ for } M=\begin{pmatrix}\alpha & -\beta\\ -\lambda & \delta\end{pmatrix}\in\Gamma_B$$

**Therem 5:** $\Delta(\tau)=(2\pi)^{12}\eta^{24}(\tau)=(2\pi)^{12}x\prod_{n=1}^{\infty}(1-x^n)^{24}$

**Proof:** Let $f(\tau)=\Delta(\tau)/\eta^{24}(\tau)$. Then $f(\tau+1)=f(\tau)$ and $f(-\frac{1}{\tau})=f(\tau)$, so $f$ is invariant under every transformation in $\Gamma$. Also, $f$ is analytic and non-zero in ℵ because $\Delta(\tau)$ is analytic and non-zero and $\eta(\tau)$ never vanishes in ℵ. Next we examine the behavior of at $i\infty$. We have

$$\eta^{24}(\tau)=e^{2\pi i\tau}\prod_{n=1}^{\infty}\left(1-e^{2\pi in\tau}\right)^{24}$$

$$=x\prod_{n=1}^{\infty}\left(1-x^n\right)^{24}=x\left(1+I(x)\right)$$

where $I(x)$ denotes a power series in $x$ with integer coefficients. Thus, $\eta^{24}(\tau)$ has a first order zero at $x=0$ [8].

At first we see the infinite products

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)$$

$$=\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau+2\pi iu}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau-2\pi iu}).$$

which it converges absolutely.

**Theorem 6:** We have the relations

$$\eta(u)=e^{\frac{\pi iu}{12}}.\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2},3u+2k)$$

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(\tfrac{u+4}{4},\tfrac{3}{2}u)=e^{-\frac{\pi iu}{12}}\eta(u).\prod_{n=1}^{\infty}(1-e^{(2n-1)\pi iu})$$

between the functions $\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)$, $\theta\begin{bmatrix}0\\1\end{bmatrix}(u,\tau)$ and Dedekind's $\eta$-function which defined by the infinite product

$$\eta(\tau)=e^{\frac{\pi i\tau}{12}}.\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau})$$

where $\operatorname{Im}\tau>0$ and k is a integer.

**Proof**

a. Let us recall the formula

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(u,\tau)$$

$$=\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau+2\pi iu}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i\tau-2\pi iu}).$$

If k integer, then we have

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2},3u+2k)$$

$$=\prod_{n=1}^{\infty}(1-e^{2n\pi i(3u+2k)}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i(3u+2k)+2\pi i(\frac{u+1}{2})}).\prod_{n=1}^{\infty}(1+e^{(2n-1)\pi i(3u+2k)-2\pi i(\frac{u+1}{2})})$$

$$=\prod_{n=1}^{\infty}(1-e^{6n\pi iu}).\prod_{n=1}^{\infty}(1+e^{6n\pi iu-2\pi iu-(2k-1)\pi i}).\prod_{n=1}^{\infty}(1+e^{6n\pi iu-4\pi iu-(2k+1)\pi i})$$

$$=\prod_{n=1}^{\infty}(1-e^{6n\pi iu}).\prod_{n=1}^{\infty}(1-e^{6n\pi iu-2\pi iu}).\prod_{n=1}^{\infty}(1-e^{6n\pi iu-4\pi iu}).$$

If we set R= $e^{2\pi iu}$ , then we obtain

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2},3u+2k)=\prod_{n=1}^{\infty}(1-R^{3n}).\prod_{n=1}^{\infty}(1-R^{3n-1}).\prod_{n=1}^{\infty}(1-R^{3n-2})\cdot$$

On the other hand, we may set $n=n'+1$, then

$$\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2},3u+2k)=\prod_{n'=1}^{\infty}(1-R^{3n'+3}).\prod_{n'=1}^{\infty}(1-R^{3n'+2}).\prod_{n'=1}^{\infty}(1-R^{3n'+1})$$

$$=(1-R)(1-R^2)(1-R^3)(1-R^4)\ldots=$$

$$\prod_{m=1}^{\infty}(1-R^m)=\prod_{m=1}^{\infty}(1-e^{2m\pi iu})$$

According to above, we have

$$\eta(\tau)=e^{\frac{\pi i\tau}{12}}.\theta\begin{bmatrix}0\\0\end{bmatrix}(\tfrac{u+1}{2},3u+2k)$$

from the Dedekind's $\eta$-function defined by the infinite product

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}}.\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau})$$

where m = $n'$.

b. According to the equation,

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(u,\tau) = \sum_{n=-\infty}^{\infty}(-1)^n \exp(n^2\pi i\tau + 2n\pi iu)$$

we have

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(\tfrac{u+4}{4},\tfrac{3}{2}u) = \sum_{n=-\infty}^{\infty}(-1)^n \exp\left[\tfrac{1}{2}n(3n+1)\pi iu\right]$$

$$=1+\sum_{n=1}^{\infty}(-1)^n\left\{\exp\left[\tfrac{1}{2}n(3n-1)\pi iu\right]+\exp\left[\tfrac{1}{2}n(3n+1)\pi iu\right]\right\}$$

$$=1+\sum_{n=1}^{\infty}(-1)^n\left[x^{\frac{1}{2}n(3n-1)}+x^{\frac{1}{2}n(3n+1)}\right]=1-$$

$$x-x^2+x^5+x^7-x^{12}-x^{15}+...$$

$$\theta\begin{bmatrix}0\\1\end{bmatrix}(\tfrac{u+4}{4},\tfrac{3}{2}u) = (1-x)(1-x^2)(1-x^3)... = \prod_{n=1}^{\infty}(1-x^n)$$

where $x=e^{\pi iu}$ for $|x|<1$ ,and $\frac{1}{2}n(3n+1)$ are known as the pentagonal numbers n= -1,-2,...

This results play a role of key stone in the forthcoming work concerning relation between the $\theta$-theta function and Dedekind's $\eta$-function. In fact, if the application of theorem(6-b) on the relation obtained with the theorem(-a) which known as the equation between Dedekind's $\eta$-function and L.Euler's theorem on pentagonal numbers is done, we obtain

$$\frac{\theta\begin{bmatrix}0\\1\end{bmatrix}(\tfrac{u+4}{4},\tfrac{3}{2}u)}{\prod_{n=1}^{\infty}\left[1-e^{(2n-1)\pi i\tau}\right]} = \frac{\sum_{n=-\infty}^{\infty}(-1)^n\exp\left[\frac{1}{2}n(3n+1)\pi iu\right]}{\prod_{n=1}^{\infty}\left[1-e^{(2n-1)\pi i\tau}\right]}$$

$$=\frac{\prod_{n=1}^{\infty}\left[1-e^{n\pi i\tau}\right]}{\prod_{n=1}^{\infty}\left[1-e^{(2n-1)\pi i\tau}\right]} = \prod_{n=1}^{\infty}\left[1-e^{2n\pi i\tau}\right] = e^{-\frac{\pi i\tau}{12}}.\eta(\tau)$$

As a result, the relation has been obtained between theta and Dedkind's -$\eta(\tau)$ functions by using the characteristic $\begin{bmatrix}0\\1\end{bmatrix}$ and the variable $\frac{u+4}{4}$ instead of the characteristic $\begin{bmatrix}0\\0\end{bmatrix}$ and the variable $\frac{u+1}{2}$ which were previously used by Jaccobi.

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