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On Parametric p-Valent Meromorphic Functions

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Abstract: Problem statement: In this research, we studied parametric p-valent meromorphic functions $P^{\alpha}f$ by considering two classes $M_p^{\alpha}(\beta)$ and $M_p^{\alpha}(\lambda, A)$. **Approach:** With the help of Jack's Lemma an inclusion relation for the class M_p^{α} was obtained and it is shown that this class is closed by an integral operator I_c. **Results:** A subordination result for the class $M_p^{\alpha}(\lambda, A)$ was proved. Consequences of main results with the results for special values of the parameter α were discussed. **Conclusion/Recommendations:** Our results certainly generalized several results obtained earlier as well as generate new results.

Key words: Meromorphic functions, starlike functions, convolution, generalized hypergeometrc functions

INTRODUCTION

Let M_p denotes a class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p}, p \in N = \{1, 2, 3, ...\} \tag{1a}$$

which are analytic and p-valent in the punctured unit disk $U^* = \{z: 0<|z|<1\} = U \backslash \{0\}$. We say that a function $f(z) \in M_p$ is in the class $M^*_p(\beta)$ if $f(z) \neq 0$ and:

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < -\beta, 0 \le \beta < p, z \in U$$

Functions in the class $M^*_p(\beta)$ are called p-valent meromorphic starlike of order β.

Let $g_{\alpha}(z) \in M_p$ be of the form:

$$g_{\alpha}(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p}([\alpha]) z^{k-p} \tag{1b}$$

whose coefficient $b_{k-p}([\alpha])$ has a parameter α which is either -p or a positive real and it satisfies the relation:

$$b_{k-p}([\alpha+1]) = \left(\frac{\alpha+k}{\alpha}\right) b_{k-p}([\alpha]) \tag{1c}$$

For $g_{\alpha}(z)$ given by (1b), a parametric convolution operator $P^{\alpha}: M_p \to M_p$, on the function f(z) of the form (1a) is defined by:

$$\begin{aligned} &P^{\alpha}f(z) \\ &= (f * g_{\alpha})(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p}([\alpha]) z^{k-p} \\ &= (g_{\alpha} * f)(z) \end{aligned} \tag{1d}$$

where, '*' stands for convolution or Hadamard product. We have:

$$P^{\alpha+1}f(z) = z^{-p} + \sum_{k=1}^{\infty} \left(\frac{\alpha+k}{\alpha}\right) a_{k-p} b_{k-p}([\alpha]) z^{k-p}$$

$$\text{For } \alpha = -p,\ P^{\alpha+1}f(z) \equiv -\frac{z(P^{\alpha}f(z))'}{p}$$

Using (1c) and (1d), we can easily obtain the identity related to parametric p-valent meromorphic functions:

$$z(P^{\alpha}f(z))' = \alpha P^{\alpha+1}f(z) - (\alpha+p)P^{\alpha}f(z) \tag{1e}$$

Several subclasses of p-valent meromorphic functions involving various convolution operators have been defined and studied in (Aouf, 2008; Liu and Srivastava, 2001; 2004; Raina and Srivastava, 2006; Srivastava and Patel, 2006; Srivastava *et al*., 2008; Wang *et al*., 2009; Yang, 2001). The purpose of this study is to unify the results obtained earlier and to give some new results. Motivated with these earlier works especially the work of Cho (Srivastava and Owa, 1992;

Yang, 2004), in this study, in order to study parametric p-valent meromorphic functions $P^{\alpha}f(z)$ of the form (1d), we consider classes $M_p^{\alpha}(\beta)$ and $M_p^{\alpha}(\lambda, A)$ of $f(z)\in M_p$ satisfying following conditions respectively:

$$\mathrm{Re}\left\{\frac{\alpha P^{\alpha+1}f(z)}{P^{\alpha}f(z)}\right\} < (\alpha+p-\beta), z\in U \quad (1f)$$

and for $\lambda>p, |A|<|\alpha|$:

$$(1-\lambda)\alpha z^p P^{\alpha}f(z) + \lambda\alpha z^p P^{\alpha+1}f(z) \prec \alpha + \left(1+\frac{\lambda}{\alpha}\right)Az, z\in U \quad (1g)$$

where, '$\prec$' stands for subordination between two analytic functions in U.

We say $f\prec g$, if there exists a Schwartz function $\omega(z)$, which is analytic in U with $\omega(0) = 0$ and $|\omega(z)|<1$, $z\in U$ such that $f(z) = g(\omega(z))$, $z\in U$. Indeed it is known that $f(z) \prec g(z)$, $z\in U \Rightarrow f(0) = g(0)$ and $f(U)\subset g(U)$.

Clearly, if $\alpha = -p$, the classes $M_p^{\alpha}(\beta)$ and $M_p^{\alpha+1}(\beta)$ are respectively the classes of p-valent meromorphic starlike and convex functions $P^{\alpha}f(z)$ of order $\beta(0\le\beta<p)$ if $P^{\alpha}f(z)$, $P^{\alpha+1}f(z)\neq 0$. We denote $M_p^{\alpha}(0)\equiv M_p^{\alpha}$.

MATERIALS AND METHODS

To prove our main results, we need following Lemmas:

Lemma 1: Let q(z) be univalent in the unit disk U and θ and φ be analytic in a domain E containing q(U) with $\varphi(w)\neq 0$ when $w\in q(U)$ (Eenigenburg *et al*., 1984). Set:

$$Q(z) := zq'(z)\varphi(q(z)),\ h(z) := \theta(q(z)) + Q(z)$$

and suppose that either Q(z) is starlike or h(z) is convex in U. In addition, assume that:

$$\mathrm{Re}\frac{zh'(z)}{Q(z)} > 0, z\in U$$

If p(z) is analytic in U with p(0) = q(0), $p(U)\subseteq E$ and:

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z))$$

Then $p(z) \prec q(z)$ and q(z) is the best dominant.

Lemma 2: Let the (non-constant) function $\omega(z)$ be analytic in U with $\omega(0) = 0$ Jack (1971) Lemma. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r<1$ at a point $z_0\in U$, then $z_0\omega'(z_0) = \gamma\omega(z_0)$, where γ is a real number and $\gamma\ge 1$.

Theorem 1: For a parametric convolution operator P^{α} defined in (1d), $M_p^{\alpha+1}\subset M_p^{\alpha}$.

Proof: Let $f\in M_p^{\alpha+1}$, we have:

$$\mathrm{Re}\left\{\frac{(\alpha+1)P^{\alpha+2}f(z)}{\alpha P^{\alpha+1}f(z)}\right\} < (\alpha+1+p) \quad (2a)$$

Define an analytic function $\omega(z)$ in U such that $\omega(0) = 0$, by:

$$\frac{\alpha P^{\alpha+1}f(z)}{P^{\alpha}f(z)} = (\alpha+p) - p\left\{\frac{1-\omega(z)}{1+\omega(z)}\right\} = \frac{\alpha+(\alpha+2p)\omega(z)}{1+\omega(z)} \quad (2b)$$

To show:

$$\mathrm{Re}\left\{\frac{\alpha P^{\alpha+1}f(z)}{P^{\alpha}f(z)}\right\} < (\alpha+p)$$

We need to show $\mathrm{Re}\left\{\frac{1-\omega(z)}{1+\omega(z)}\right\} > 0$ or, $|\omega(z)|<1$ in U. Differentiating logarithmically (2b), we get:

$$\begin{aligned}\frac{z\left(P^{\alpha+1}f(z)\right)'}{P^{\alpha+1}f(z)} - \frac{z\left(P^{\alpha}f(z)\right)'}{P^{\alpha}f(z)} &= \frac{(\alpha+2p)z\omega'(z)}{\alpha+(\alpha+2p)\omega(z)} \\ -\frac{z\omega'(z)}{1+\omega(z)} &= \frac{2pz\omega'(z)}{\left[\alpha+(\alpha+2p)\omega(z)\right][1+\omega(z)]}\end{aligned} \quad (2c)$$

Applying the identity (1e) for α and $\alpha+1$ in (2c) and using (2b), we get:

$$\frac{(\alpha+1)P^{\alpha+2}f(z)}{P^{\alpha+1}f(z)} = \alpha+1+p-p\left\{\frac{1-\omega(z)}{1+\omega(z)}\right\} + \frac{2pz\omega'(z)}{\left[\alpha+(\alpha+2p)\omega(z)\right][1+\omega(z)]}$$

Let there exists a point $z_0\in U$ such that $|\omega(z_0)| = 1$, then by lemma 2, $z_0\omega'(z_0) = \gamma\omega(z_0)$, $\omega(z_0) = e^{i\theta}$, $\gamma\ge 1$. Therefore:

$$\frac{(\alpha+1)P^{\alpha+2}f(z_0)}{P^{\alpha+1}f(z_0)}=\alpha+1+p-p\left\{\frac{1-\omega(z_0)}{1+\omega(z_0)}\right\}$$
$$+\frac{2pz_0\omega'(z_0)}{\left[\alpha+(\alpha+2p)\omega(z_0)\right][1+\omega(z_0)]}$$
$$=\alpha+1+p-p\left\{\frac{1-e^{i\theta}}{1+e^{i\theta}}\right\}+\frac{2p\gamma e^{i\theta}}{\left[\alpha+(\alpha+2p)e^{i\theta}\right]\left[1+e^{i\theta}\right]}$$

Hence:

$$\operatorname{Re}\left\{\frac{(\alpha+1)P^{\alpha+2}f(z)}{P^{\alpha+1}f(z)}\right\}=(\alpha+1+p)$$
$$+\operatorname{Re}\left\{\frac{2p\gamma e^{i\theta}}{\left[\alpha+(\alpha+2p)e^{i\theta}\right]\left[1+e^{i\theta}\right]}\right\}$$
$$\geq(\alpha+1+p),\ \alpha=-p \text{ or } \alpha>0$$

which contradicts (2a), this proves that $|\omega(z)|<1$ in U and hence Theorem 1 is proved.

Taking $\alpha = -p$, in Theorem 1, we get following result.

Corollary 1: Let $P^{\alpha}f(z)\in M_p$ be defined in (1d) with $\alpha=-p$ and $P^{\alpha}f(z),\ z(P^{\alpha}f(z))'\neq 0$, if:

$$\operatorname{Re}\left\{1+\frac{z\left(P^{\alpha}f(z)\right)''}{\left(P^{\alpha}f(z)\right)'}\right\}<0$$

Then:

$$\operatorname{Re}\left\{\frac{z\left(P^{\alpha}f(z)\right)'}{P^{\alpha}f(z)}\right\}<0$$

Theorem 2: Let an integral operator I_c: $M_p\to M_p$ be defined for c>0 by:

$$I_cf(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\,dt \tag{2d}$$

$$=z^{-p}+\sum_{k=1}^{\infty}\left(\frac{c}{c+k}\right)a_{k-p}z^{k-p} \tag{2e}$$

The class M_p^{α}, defined in (1f) is closed under the integral operator I_c.

Proof: Let $f\in M_p^{\alpha}$, we have:

$$\operatorname{Re}\left\{\frac{\alpha P^{\alpha+1}f(z)}{P^{\alpha}f(z)}\right\}<(\alpha+p) \tag{2f}$$

Define an analytic function $\omega(z)$ in U such that $\omega(0) = 0$, by:

$$\frac{\alpha P^{\alpha+1}I_cf(z)}{P^{\alpha}I_cf(z)}=(\alpha+p)-p\left\{\frac{1-\omega(z)}{1+\omega(z)}\right\} \tag{2g}$$

$$=\frac{\alpha+(\alpha+2p)\omega(z)}{1+\omega(z)} \tag{2h}$$

To show:

$$\operatorname{Re}\left\{\frac{\alpha P^{\alpha+1}I_cf(z)}{P^{\alpha}I_cf(z)}\right\}<(\alpha+p)$$

We need to show $\operatorname{Re}\left\{\frac{1-\omega(z)}{1+\omega(z)}\right\}>0$ or, $|\omega(z)<1$ in U. Differentiating logarithmically (2h), we get:

$$\frac{z\left(P^{\alpha+1}I_cf(z)\right)'}{P^{\alpha+1}I_cf(z)}-\frac{z\left(P^{\alpha}I_cf(z)\right)'}{P^{\alpha}I_cf(z)}=\frac{(\alpha+2p)z\omega'(z)}{\alpha+(\alpha+2p)\omega(z)}$$
$$-\frac{z\omega'(z)}{1+\omega(z)}=\frac{2pz\omega'(z)}{\left[\alpha+(\alpha+2p)\omega(z)\right][1+\omega(z)]} \tag{2i}$$

Using the series expansion of $I_cf(z)$ given in (2e), we get the identity:

$$z(P^{\alpha}I_cf(z))'=cP^{\alpha}f(z)-(c+p)P^{\alpha}I_cf(z) \tag{2j}$$

Applying the identity (2j) for α and $\alpha+1$ in (1e) for $I_cf(z)$ in (2i) and using (2g), we get:

$$\frac{\alpha P^{\alpha+1}f(z)}{P^{\alpha}f(z)}=\frac{\alpha P^{\alpha+1}I_cf(z)}{P^{\alpha}I_cf(z)}$$
$$\left[1+\frac{\left\{\frac{z(P^{\alpha+1}I_cf(z))'}{P^{\alpha+1}I_cf(z)}-\frac{z(P^{\alpha}I_cf(z))'}{P^{\alpha}I_cf(z)}\right\}}{\frac{z(P^{\alpha}I_cf(z))'}{P^{\alpha}I_cf(z)}+(c+p)}\right]=\alpha+p-p$$
$$\left\{\frac{1-\omega(z)}{1+\omega(z)}\right\}+\frac{2pz\omega'(z)}{\left[\alpha+(\alpha+2p)\omega(z)\right][1+\omega(z)]}$$

Let there exists a point $z_0\in U$ such that $|\omega(z_0)| = 1$, then by lemma 2, $z_0\omega'(z_0) = \gamma\omega(z_0)$, $\omega(z_0) = e^{i\theta}$, $\gamma\geq 1$. Therefore:

$$\frac{\alpha P^{\alpha+1} f(z_0)}{P^{\alpha} f(z_0)} = \alpha + p - p\left\{\frac{1-\omega(z_0)}{1+\omega(z_0)}\right\}$$
$$= (\alpha+p) - p\left\{\frac{1-e^{i\theta}}{1+e^{i\theta}}\right\} + \frac{2p\gamma e^{i\theta}}{\left[c+(c+2p)e^{i\theta}\right]\left[1+e^{i\theta}\right]}$$

Hence:

$$\mathrm{Re}\left\{\frac{\alpha P^{\alpha+1} f(z)}{P^{\alpha} f(z)}\right\} = (\alpha+p) + \mathrm{Re}\frac{2p\gamma e^{i\theta}}{\left[\alpha+(\alpha+2p)e^{i\theta}\right]\left[1+e^{i\theta}\right]}$$
$$\geq (\alpha+p), \alpha = -p \text{ or } \alpha > 0$$

which contradicts (2f), this proves that $|\omega(z)|<1$ in U and hence Theorem 2 is proved.

Remark: The results obtained in Theorems 1 and 2, coincide for $\alpha = n + p>0$, with the results obtained by Cho in (Srivastava and Owa, 1992).

Theorem 3: Let $f(z)\in M_p$ be in the class $M_p^{\alpha}(\lambda, A)$, defined in (1g), then $\alpha z^p P^{\alpha} f(z) \prec \alpha + Az$.

Proof: Let $p(z) = \alpha z^p P^{\alpha} f(z)$, we have $p(z) + \frac{\lambda}{\alpha} zp'(z) \prec \alpha + \left(1+\frac{\lambda}{\alpha}\right)Az$. Consider $q(z) = \alpha+Az$ which is univalent, convex in U. Consider for $\lambda>p$:

$$\theta(w) = w, \varphi(w) = \frac{\lambda}{\alpha}, (w \in C)$$

which are analytic in C so that:

$$\theta(p(z)) + zp'(z)\varphi(p(z)) = p(z) + \frac{\lambda}{\alpha} zp'(z)$$

Set:

$$Q(z) := zq'(z)\varphi(q(z)),\ h(z) := \theta(q(z)) + Q(z)$$

We obtain that:

$$Q(z) = \frac{\lambda}{\alpha} zq'(z)$$

which is starlike in U and:

$$\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathrm{Re}\left\{\frac{\alpha}{\lambda} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0, \alpha = -p \text{ or } \alpha > 0$$

Hence on applying Lemma 1, we get that $p(z)\prec q(z)$ or $\alpha z^p P^{\alpha} f(z) \prec \alpha+Az$, $\alpha+Az$ is the best dominant and the result is sharp with extremal function:

$$P^{\alpha} f(z) = z^{-p} \exp\left\{\int_0^z \frac{A}{(\alpha+At)} dt\right\}$$

Corollary 2: Let $f(z)\in M_p$ be in the class $M_p^{\alpha}(\lambda, A)$, defined in (1g), then $\left|\alpha z^p P^{\alpha} f(z)\right| > |\alpha| - |A|$.

Corollary 3: Let $f(z)\in M_p$ be in the class $M_p^{\alpha}(\lambda, A)$, defined in (1g), then $\sum_{k=1}^{\infty}\left|a_{k-p} b_{k-p}([\alpha])\right| \leq \left|\frac{A}{\alpha}\right|$. The result is sharp for the extremal function given by:

$$f_k(z) = z^{-p} + \left|\frac{A}{\alpha}\right| \frac{1}{\left|b_{k-p}([\alpha])\right|} z^{k-p}, k \geq 1$$

RESULTS AND DISCUSSION

Some consequences of the results are discussed along with some special cases as follows:

For positive real numbers a_i, $b_i (i = 1,2,\ldots,q)$ and for positive integers A_i, $B_i (i = 1,2,\ldots,q)$ such that $\sum_{i=1}^{q}(B_i - A_i) \geq 0$, taking:

$$b_{k-p}([\alpha]) = b_{k-p}([a_1, A_1]) := \prod_{i=1}^{q} \frac{\Gamma(a_i + A_i k)\Gamma(b_i)}{\Gamma(b_i + B_i k)\Gamma(a_i)}$$

We get for $f(z)\in M_p$ and for $\alpha>0$:

$$P^{\alpha} f \equiv W([a_1, A_1]) f := z^{-p} \prod_{i=1}^{q} \frac{\Gamma(b_i)}{\Gamma(a_i)}\, {}_{q+1}\Psi_q(z) * f \tag{3a}$$

where, ${}_{q+1}\psi_q(z)$ is Wright's (psi) function which is a generalized hypergeometric function (Srivastava and Manocha, 1984) and its series representation is given by:

$${}_{q+1}\Psi_q\left((a_1, A_1), \ldots (a_q, A_q); (b_1, B_1), \ldots (b_q, B_q); z\right)$$
$$= \sum_{k=o}^{\infty} \prod_{i=1}^{q} \frac{\Gamma(a_i + A_i k)}{\Gamma(b_i + B_i k)} z^k$$

and:

$$b_{k-p}([\alpha+1]) = b_{k-p}([a_1+1, A_1])$$
$$:= \frac{(a_1 + A_1 k)}{a_1} b_{k-p}([a_1+1, A_1]) \tag{3b}$$

Hence, from:

$$W([a_1,A_1])f := W((a_1,A_1),...(a_q,A_q),(b_1,B_1),...(b_q,B_q))f$$

and:

$$W([a_1+1,A_1])f := W\begin{pmatrix}(a_1+1,A_1),...(a_q,A_q),\\(b_1,B_1),...(b_q,B_q)\end{pmatrix}f$$

with the help of relation (3b), we can easily get the identity:

$$A_1z\left(W([a_1,A_1])f\right)' = a_1W([a_1+1,A_1])f \\ -(a_1+pA_1)W([a_1,A_1])f$$

which can directly be obtained by taking $\alpha = \frac{a_1}{A_1}$ in (1e). The operator W($[a_1, A_1]$)f, is itself a generalized operator and is a meromorphic version of the operator considered in (Dziok and Raina, 2004; 2009; Aouf and Dziok, 2008a; 2008b; Sharma, 2010; Dziok *et al*., 2004). Taking $A_i = B_i = 1$, i = 1,2,…,q, ${}_{q+1}\psi_q(z)$ reduces to the generalized hypergeometric function ${}_{q+1}F_q$ and we denote:

$$P^{\alpha}f \equiv F([a_1])f := z^{-p}\,{}_{q+1}F_q(z) * f \qquad (3c)$$

Where:

$${}_{q+1}F_q(z) := {}_{q+1}F_q(a_1,...a_q,1;b_1,...b_q;z)$$

The operator similar to F($[a_1]$)f, has been studied recently by Wang *et al*. (2009); Raina and Srivastava (2006); Aouf (2008) and Liu and Srivastava (2004). Taking q = 2 and $b_2 = 1$, we get the operator, involving Gauss's hypergeometric function ${}_2F_1$:

$$P^{\alpha}f \equiv G(a_1,a_2,b_1)f := z^{-p}\,{}_2F_1(a_1,a_2;b_1;z) * f \qquad (3d)$$

Further, taking q = 1, we get:

$$P^{\alpha}f \equiv L_p(a_1,b_1)f := z^{-p}\,{}_2F_1(a_1,1;b_1;z) * f \qquad (3e)$$

which is studied extensively by Liu and Srivastava (2001); Liu (2000); Aouf and Srivastava (2006) and Srivastava *et al*. (2008). If $a_1 = n + p$, $n\in N$ and $b_1 = 1$, we get, Ruscheweyh derivative operator for M_p class:

$$P^{\alpha}f \equiv D^{n+p-1}f := \frac{z^{-p}}{(1-z)^{n+p}} * f \qquad (3f)$$

which is studied by Cho in (Srivastava and Owa, 1992; Uralegaddi and Somanatha, 1992; Joshi and Srivastava, 1999; Aouf, 1993) and its special case is studied by Aouf and Hossen (1994).

From Theorem 1, we directly get following results.

Corollary 4: Let for $f(z)\in M_p$ the operator W(a_1,A_1])f be defined in (3a). If:

$$\mathrm{Re}\left\{\frac{W([a_1+2,A_1])f}{W([a_1+1,A_1])f}\right\} < \frac{(a_1+1+pA_1)}{(a_1+1)}$$

Then:

$$\mathrm{Re}\left\{\frac{W([a_1+1,A_1])f}{W([a_1,A_1])f}\right\} < \frac{(a_1+pA_1)}{a_1}$$

Corollary 5: Let for $f(z)\in M_p$ the operator W($[a_1]$)f be defined in (3c). If:

$$\mathrm{Re}\left\{\frac{F([a_1+2])f}{F([a_1+1])f}\right\} < \frac{(a_1+1+p)}{(a_1+1)}$$

Then:

$$\mathrm{Re}\left\{\frac{F([a_1+1])f}{F([a_1])f}\right\} < \frac{(a_1+p)}{a_1}$$

Corollary 6: Let for $f(z)\in M_p$ the operator G(a_1, a_2, b_1)f be defined in (3d). If:

$$\mathrm{Re}\left\{\frac{G(a_1+2,a_2,b_1)f}{G(a_1+1,a_2,b_1)f}\right\} < \frac{(a_1+1+p)}{(a_1+1)}$$

Then:

$$\mathrm{Re}\left\{\frac{G(a_1+1,a_2,b_1)f}{G(a_1,a_2,b_1)f}\right\} < \frac{(a_1+p)}{a_1}$$

Corollary 7: Let for $f(z)\in M_p$the operator $L_p(a_1, b_1)$f be defined in (3e). If:

$$\mathrm{Re}\left\{\frac{L_p(a_1+2,b_1)f}{L_p(a_1+1,b_1)f}\right\} < \frac{(a_1+1+p)}{(a_1+1)}$$

Then:

$$\mathrm{Re}\left\{\frac{L_p(a_1+1,b_1)f}{L_p(a_1,b_1)f}\right\} < \frac{(a_1+p)}{a_1}$$

Corollary 8: Let for $f(z)\in M_p$ the operator $D^{n+p-1}f$ be defined in (3f) (Srivastava and Owa, 1992). If:

$$\mathrm{Re}\left\{\frac{D^{n+p+2}f}{D^{n+p}f}\right\} < \frac{(n+1+2p)}{(n+1+p)}$$

Then:

$$\mathrm{Re}\left\{\frac{D^{n+p}f}{D^{n+p-1}f}\right\} < \frac{(n+2p)}{(n+p)}$$

Corollary 9: Let the operator I_c be defined by (2d) and the parametric class M_p^{α} be defined in (1f), if $I_c f\in M_p^{\alpha+1}$, then $I_c^2 f\in M_p^{\alpha}$.

Proof: By Theorem 2, if $I_c f\in M_p^{\alpha+1}$, $I_c I_c f = I_c^2 f\in M_p^{\alpha+1}$ and further, by Theorem 1, we get the result.

Corollary 10: Let the parameter α be positive in the definition of parametric class M_p^{α} defined in (1f):

$$f\in M_p^{\alpha} \Leftrightarrow I_\alpha f\in M_p^{\alpha+1}$$

Proof: Taking c = $\alpha>0$, we see from (1d) and (2d), that:

$$P^{\alpha+1} I_\alpha f(z) = P^{\alpha} f(z)$$

This directly proves the result.

From Theorem 2, we get following special case:

Corollary 11: Let for $f(z)\in M_p$, the operators $W([a_1,A_1])f$ and $I_c f$ be defined by (3a) and (2d) respectively. If:

$$\mathrm{Re}\left\{\frac{W([a_1+1,A_1])f}{W([a_1,A_1])f}\right\} < \frac{(a_1+pA_1)}{a_1}$$

Then:

$$\mathrm{Re}\left\{\frac{W([a_1+1,A_1])I_c f}{W([a_1,A_1])I_c f}\right\} < \frac{(a_1+pA_1)}{a_1}$$

Similar to Corollaries 5-8, we can further find special results of the Corollary 11.

Convolution condition follows for the class $M_p^{\alpha}(\lambda,A)$.

Corollary 12: Let for $\lambda>p$, $|A|<|\alpha|$, $f(z)\in M_p^{\alpha}(\lambda,A)$ if and only if:

$$\alpha+\left(1+\frac{\lambda}{\alpha}\right)Ae^{i\theta} - z^p f * \left[\alpha+\sum_{k=1}^{\infty}(\alpha+\lambda k)a_{k-p}b_{k-p}([\alpha])z^k\right]$$
$$\neq 0, z\in U, 0\le\theta\le 2\pi$$

From Theorem 3, we get following results.

Corollary 13: For $p<\lambda_1<\lambda_2$, $|A|<|\alpha|$:

$$M_p^{\alpha}(\lambda_2,A)\subset M_p^{\alpha}(\lambda_1,A)$$

Proof: Let $f(z)\in M_p^{\alpha}(\lambda_2,A)$. Consider:

$$(1-\lambda_1)\alpha z^p P^{\alpha} f(z) + \lambda_1 \alpha z^p P^{\alpha+1} f(z) = \left(1-\frac{\lambda_1}{\lambda_2}\right)\alpha z^p P^{\alpha} f(z)$$
$$+\frac{\lambda_1}{\lambda_2}\left[(1-\lambda_2)\alpha z^p P^{\alpha} f(z) + \lambda_2\alpha z^p P^{\alpha+1} f(z)\right] \prec \alpha+\left(1+\frac{\lambda}{\alpha}\right)Az$$

On using the given hypothesis and the result of Theorem 3. This proves the result.

Corollary14: For $p<\lambda<\frac{2p^2}{|A|}$, $|A|<|\alpha|$:

$$M_p^{\alpha}(\lambda,A)\subset M_p^{\alpha}(\beta)$$

Where:

$$\beta = p-\frac{1}{\lambda}\left(\frac{2|\alpha|+\frac{\lambda}{\alpha}|A|}{1-\left|\frac{A}{\alpha}\right|}\right)$$

Proof: With the given hypothesis and from Corollary 2 of Theorem 3, we get:

$$\left|\alpha z^p P^{\alpha+1} f(z) - \alpha z^p P^{\alpha} f(z)\right| = \left|\frac{1}{\lambda}\left[\begin{array}{l}(1-\lambda)\alpha z^p P^{\alpha} f(z) + \\ \lambda\alpha z^p P^{\alpha+1} f(z) - \alpha z^p P^{\alpha} f(z\end{array}\right]\right|$$
$$\le \frac{1}{\lambda}\left(\frac{2|\alpha|+\frac{\lambda}{\alpha}|A|}{1-\left|\frac{A}{\alpha}\right|}\right)\left|z^p P^{\alpha} f(z)\right|$$

Hence:

$$\left|\frac{\alpha P^{\alpha+1}f(z)}{P^{\alpha}f(z)}-\alpha\right|<\frac{1}{\lambda}\left(\frac{2|\alpha|+\dfrac{\lambda}{\alpha}|A|}{1-\left|\dfrac{A}{\alpha}\right|}\right)$$

Or:

$$\operatorname{Re}\left\{\frac{\alpha P^{\alpha+1}f(z)}{P^{\alpha}f(z)}\right\}<\alpha+p-\beta$$

Where:

$$\beta=p-\frac{1}{\lambda}\left(\frac{2|\alpha|+\dfrac{\lambda}{\alpha}|A|}{1-\left|\dfrac{A}{\alpha}\right|}\right)$$

which proves the result.

Corollary 15: Let $f(z)\in M_p^{\alpha}(\lambda,A)$ defined in (1g), then $I_1f(z)\in M_p^{\alpha}(p-1)$, where:

$$I_1f(z)=\frac{1}{z^{p+1}}\int_0^z t^p f(t)dt$$

Proof: From Corollary 3 of Theorem 3, we get:

$$\sum_{k=1}^{\infty}\left|a_{k-p}b_{k-p}([\alpha])\right|\le\left|\frac{A}{\alpha}\right|<1$$

Therefore:

$$\left|\frac{\alpha P^{\alpha+1}I_1f(z)}{P^{\alpha}I_1f(z)}-\alpha\right|\le\frac{\sum_{k=1}^{\infty}\frac{k}{k+1}\left|a_{k-p}b_{k-p}([\alpha])\right|}{1-\sum_{k=1}^{\infty}\frac{1}{k+1}\left|a_{k-p}b_{k-p}([\alpha])\right|}<1$$

which proves the result.

Results for suitable values of parameter α, based on Corollaries 12-15 can also be obtained.

CONCLUSION

An inclusion relation for the class M_p^{α} is obtained and it is shown that this class is preserved under the operator I_c (c>0). The subordinate condition for the class $M_p^{\alpha}(\lambda,A)$ provides a convolution condition and a subordination which gives an inclusion relation, a sharp coefficient inequality. It is verified that $M_p^{\alpha}(\lambda,A)\subset M_p^{\alpha}(\beta)$ for some β and the operator $I_1: M_p^{\alpha}(\lambda,A)\to M_p^{\alpha}(p-1)$. The results verify the inclusions related to starlike, convex and close-to-convex classes. Necessary and sufficient coefficient conditions for the classes M_p^{α} and $M_p^{\alpha}(\lambda,A)$ can also be obtained for $a_{k-p}\ge 0, k\ge 1$. With the help of coefficient conditions several more results can be derived.

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