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Modules in σ[M] with Chain Conditions on δ_M^- Small Submodules

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Abstract: Problem statement: Let M be a right module over a ring R. In this article modules in σ[M] with chain conditions on δ_M- small submodules are studied. **Approach:** With the help of known results about M- singular, Artinian and Noetherian modules the techniques of the proofs of our main results use the properties of δ_M- small, δ_M- supplement and δ_M- semimaximal submodules. **Results:** Modules in σ[M] with chain conditions on δ_M- small are investigated, δ_M- semimaximal submodule is defined . Some Properties of δ_M- semimaximal submodules are proved. As application a new characterization of Artinian module in σ[M] is obtained in terms of δ_M^- small submodules and δ_M- semimaximal submodules, as well as δ_M- small submodules and δ_M- supplement submodules. **Conclusion/Recommendations:** Our results certainly generalized several results obtained earlier.

Key words: Small submodules, supplement submodules, chain conditions, M-singular, supplemented module, finitely generated, uniform dimension, nonzero submodules, positive integer

INTRODUCTION

Throughout this research, R denotes an associative ring with unity and modules M are unitary right R-modules Mod-R denotes the category of all right R-modules. Let M be any R - module. Any R- module N is M -generated (or generated by M) if there exists an epimorphism $f: M^{(\Lambda)} \to N$, for some indexed set Λ. An R -module N is said to be subgenerated by M if Nis isomorphic to a submodule of an M -generated module. We denote by σ[M] the full subcategory of the right R-modules whose objects are all right R-modules subgenerated byM. Any module $N \in \sigma[M]$ is said to be M-singular if $N \cong L/K$, for some $L \in \sigma[M]$ and K is essential in L The class of all M-singular modules is closed under submodules, homohorphic images and direct sums. The concept of small submodule has been generalized to δ- small submodule by Zhou (2000). Zhou called a submodule N of a module M is δ- small in M (notation $N \leq_\delta M$) if, whenever N+X=M with M/X singular, we have X=M Ozcan and Alkan consider this notation in σ[M] For a module N in σ[M], Ozcan and Alkan (2006) call a submodule L of N is δ-M small submodule, written $L \ll_{\delta_M} N$, in N if L +K≠N for any proper submodule K of N with N/K M-singular. Clearly, if Lis δ- small , then L is a δ_M – small submodule.

MATERIALS AND METHODS

Hence δ_M – small submodules are the generalization of δ- small submodules in the category Mod-R Let L,K be two submodules of M L is called a δ- supplement of Kin M if M= L+K and $L \cap K \ll_\delta L$. L is called a δ- supplement submodule of M if L is a δ- supplement of some submodule of M.M is called a δ – supplemented module if every submodule of M has a δ- supplement inM. If for every submodules L,K of M with M=L+K there exists a δ – supplement N of L in Msuch that $N \leq K$, then M is called an amply δ – supplemented module. Now , let $N \in \sigma[M]$ and $L, K \leq N$. L is called a δ_M-supplement of K in N if N=K+L and $K \cap L \ll_{\delta_M} L$. L is called a δ_M-supplement submodule of N if L is a δ_M-supplement of some submodule of N Nis called a δ_M – supplemented module if every submodule of N has a δ_M – supplement. On the other hand N is called an amply δ_M – supplemented module if for every submodules L,K with N= L+K there exists a δ_M – supplement X of L such that $X \leq K$. For the other definitions and notations in this study we refer to Anderson and Fuller (1974) and Wisbauer (1991).

The properties of δ- small submodules that are listedin in Zhou (2000) Lemma 1.3 also hold in σ[M].

We write them for convenience Ozcan and Alkan, (2006) lemma 2.3, Lemma 2.1).

Lemma 1.1: Let $N \in \sigma[M]$:

1. For modulesK and L with, $K \le L \le N$, we have $L \ll_{\delta_M} N$ if and only if $K \ll_{\delta_M} N$ and $L/K \ll_{\delta_M} N/K$
2. For submodules K and L of N, $K + L \ll_{\delta_M} N$ if and only if $K \ll_{\delta_M} N$ and $L \ll_{\delta_M} N$
3. If $K \ll_{\delta_M} N, L \in \sigma[M]$ and $f: K \to L$ is a homomorphism, then $f(k) \ll_{\delta_M} L$ In particular, if $K \ll_{\delta_M} N \le L$, then $K \ll_{\delta_M} L$
4. If $K \le L \le^{\oplus} N$ and $K \ll_{\delta_M} N$, then $K \ll_{\delta_M} L$

Also Ozcan and Alkan (2006) consider the following submodule of a module N in σ[M] Zhou (2000).

$\delta_M(N) = \bigcap\{K \le N : N/K \text{ is } M\text{- singular simple }\}$

Lemma 1.2: For any N in $\sigma[M], \delta_M(N) = \sum\{L \le N : L \ll_{\delta_M} N\}$.

The next Lemma is proven in Alattass (2011).

Lemma 1.3: Let $N \in \sigma[M]$ be δ_M-supplemented. Then $N/\delta_M(N)$ is semisimple.

RESULTS AND DISCUSSION

Theorem 2.1: Let $N \in \sigma[M]$. Then $\delta_M(N)$ is Noetherian if and only if N satisfies ACC on δ_M – small submodules.

Proof: By lemma 1.2, every ascending chain of δ_M – small submodules of N is ascending chain submodules of $\delta_M(N)$. Hence the necessity is clear.

Sufficiency: Suppose to the contrary that $\delta_M(N)$ is not Noetherian. Then there is a properly ascending chain $N_1 \le N_2 \le \cdots$ of submodules of $\delta_M(N)$. Let $n_1 \in N_1$ and $n_i \in N_i - N_{i-1}$, for each $i > 1$. For each $j \ge 1$, let $K_j = \sum_{i=1}^{i=j} n_i R$. Hence K_j is finitely generated and $K_j \le \delta_M(N)$. So, by Lemma 1.2 and Lemma 1.1, $K_j \ll_{\delta_M} N$, for each $j \ge 1$. Hence $K_1 \le K_2 \le \ldots$ is a properly ascending chain of δ_M – small submodules of N. This implies N fails to satisfy ACC on δ_M – small submodules, a contradiction. Thus $\delta_M(N)$ is Noetherian.

Recall that a module M is said to have a uniform dimension n, where n is a nonnegative integer ,if n is the maximal number of summands in a direct sum of nonzero submodules of M. In this case we write u.dim M = n and we say M has a finite uniform dimension.

Theorem 2.2: For any $N \in \sigma[M]$, the following are equivalent:

a) $\delta_M(N)$ has a finite uniform dimension.
b) Every δ_M – small submodules of N has a finite uniform dimension and there exists a positive integer n such that $u.\dim L \le n$, for any $L \ll_{\delta_M} N$.
c) N does not contain an infinite direct sum of nonzero δ_M – small submodules of N

Proof: $(a) \Rightarrow (b)$. This is clear as any δ_M – small submodule of N is contained in $\delta_M(N)$.

$(b) \Rightarrow (c)$. Assume that $N_1 \oplus N_2 \oplus \cdots$ is an infinite direct sum of nonzero δ_M – small submodules of N. Then, by lemma 1.1, $N_1 \oplus N_2 \oplus \cdots \oplus N_{n+1} \ll_{\delta_M} N$ and hence $u.\dim(N_1 \oplus N_2 \oplus \cdots \oplus N_{n+1}) \ge n+1$, a contradiction to the hypothesis. Hence (C) follows.

$(c) \Rightarrow (a)$. Let $N_1 \oplus N_2 \oplus \cdots$ be an infinite direct sum of nonzero submodules of $\delta_M(N)$. For each $i \ge 1$, let n_I be a nonzero element of N_I Hence, by Lemmas 1.1 and 1.2, $n_i R \ll_{\delta_M} N$. Thus $n_1 R \oplus n_2 R \oplus \cdots$ is an infinite direct sum of nonzero δ_M – small submodules o f N This contradicts (C) and hence $\delta_M(N)$ has a finite uniform dimension.

Theorem 2.3: Let $N \in \sigma[M]$. Then the following are equivalent:

a) $\delta_M(N)$ is Artinian.
b) Every δ_M – small submodule of N is Artinian.
c) satisfies DDC on δ_M – small submodules of N

Proof: $(a) \Rightarrow (b)$. This is clear as every δ_M – small submodules of N is a submodule of $\delta_M(N)$.

$(b) \Rightarrow (c)$. This is obvious.

$(c) \Rightarrow (a)$. By Anderson and Fuller (1994), proposition 10.10) it will be suffice to show that every factor module of $\delta_M(N)$ is finitely cogenerated. For this suppose that there exists a factor module of $\delta_M(N)$ which is not finitely cogenerated. Then the set

$\Lambda = \{L \leq \delta_M(N) : \delta_M(N)/L$ is not finitely cogenerated$\}$ is nonempty . We show that Λ has a minimal member. Let $\{L_\alpha\}_{\alpha\in\Gamma}$ be a chain of submodules in Λ Consider the submodule $L = \bigcap_{\alpha\in\Gamma} L_\alpha$. If $L \notin \Lambda$, then $\delta_M(N)/L$ finitely cogenerated and so $L = L_\alpha$, for some a∈T a contradiction. This contradiction gives $L\in\Lambda$ and we conclude that every chain of Λ has a lower bound in Λ. Hence, by Zorn's lemma, Λ has a minimal member K.

We claim that $K \ll_{\delta_M} N$. First we show $Soc(\delta_M(N)/K)$ is not finitely generated. Let $x \in \delta_M(N)$ and $x \notin K$. By lemmas 1.2-1.1 , $xR \ll_{\delta_M} N$. Hence xR is Artinian. This implies $(xR+K)/K$ is a nonzero Artinian as $(xR+K)/K \cong xR/(xR\bigcap K)$. Therefore $(xR+K)/K$ and hence $\delta_M(N)/K$ has an essential socle. Thus $Soc(\delta_M(N)/K)$ is not finitely generated Anderson and Fuller (2000), Proposition 10.7.

Now suppose that U is a submodules of N such that $N = K + U$ with N/U M- singular. Let V be a submodule of $\delta_M(N)$, containing K such that $V/K = Soc(\delta_M(N)/K)$. Then we have $V = K + (U \bigcap V)$. Suppose to the contrary that $K \bigcap U \neq K$. Then $\delta_M(N)/(K\bigcap U)$ is finitely cogenerated. But $V/K \cong (K+(U\bigcap V))/K \cong (U\bigcap V)/(K\bigcap U)$ $\leq Soc(\delta_M(N)/(K\bigcap U))$. So V/K is finitely generated, a contradiction. This contradiction gives $K\bigcap U = K$ and hence N=U Thus $K \ll_{\delta_M} N$.

Next we show $V \ll_{\delta_M} N$. Suppose that $W \leq N$ such that N=V+W with N/W M- singular. Then $N/(K+W) = (U+W)/(K+W) \cong U/(K+U\bigcap W)$, implying that $N/(K+W)$ is semisimple. If N≠K+W then K+W N is contained in a maximal submodule Z of N Therefore N/Z is M- singular simple. It follows that $U \leq \delta_M(N) \leq Z$ and so N=Z, a contradiction. Thus N=K+W which will imply N=W So $V \ll_{\delta_M} N$. Therefore, by the hypothesis, V and hence V/K is Artinian.

The following example explain that if every δ_M-small submodule of N is Noetherian, then δ_M-(N) need not be Noetherian.

Example 2.4: Let $R = \mathbb{Z}, M = \mathbb{Z}$ and let $N = \mathbb{Z}_{(p^\infty)}$, the Prufer P- group. Hence N is an R- module in fact $N \in \sigma[M]$. It is known that every submodule of N is Noetherian, but N is not Noetherian. Moreover $\delta_M(N) = N$ Wang (2007), Example 2.6.

Remark: If we look to a ring R as a module over it self and taking M=R in 2.1,2.2, 2.3 we get the results 2.3, 2.4,2.5 in Wang (2007) respectively.

Recall that a submodule N of an R- module M is called a δ- semimaximal submodule if $N = \bigcap_{\alpha\in\Lambda} N_\alpha$, for some finite set Λ with $N_\alpha \leq M$ and M/N_α singular simple, for each $\alpha \in \Lambda$. Here we consider this definition in the category $\sigma[M]$.

Definition 2.5: Let $N \in \sigma[M]$ and $K \leq N$. K is called δ_M – semimaximal submodule of N if there is a finite collection $\{A_\alpha\}_{\alpha\in\Lambda}$ of submodules of N such that $K = \bigcap_{\alpha\in\Lambda} A_\alpha$ and N/A_α M- singular simple for any $\alpha \in \Lambda$.

Since any M- singular module is singular, any δ_M – semimaximal submodule of $N \in \sigma[M]$ is δ – semimaximal submodule of N. The next example gives a module with a δ – semimaximal submodule which is not δ_M – semimaximal submodue.

Example 2.6: Let M be a simple non projective module. Then M is singular and not M-singular Wisbauer (1991). Hence the trivial submodule is a δ – semimaximal submodule of M but it is not δ_M – semimaximal submodule.

Lemma 2.7: Let $N \in \sigma[M]$. Then:

1. $\delta_M(N)$ is contained in any δ_M – semimaximal submodule of N
2. If N has DDC on the δ_M – semimaximal submodules, then N has a minimal δ_M – semimaximal submodule

Proof: The proof is standard and is omitted.

Theorem 2.8: Let $N \in \sigma[M]$. Then the following statements are equivalent:

a) N is Artinian
b) N satisfies DCC on δ_M – small submodules and on δ_M – semimaximal submodules
c) N satisfies DCC on δ_M – small submodules and $\delta_M(N)$ is δ_M – semimaximal submodule
d) N amply δ_M – supplemented satisfies DCC on δ_M – small submodules and δ_M – suplementet submodules.

Proof: $(a) \Rightarrow (b)$. Is obvious.

$(b) \Rightarrow (c)$. Let K be a minimal $\delta_M -$semimaximal submodule of N. We show that $\delta_M(N) = K$.

If $\delta_M(N) = N$, then, by Lemma 2.7 (1), $N = \delta_M(N) \le K$ and so $\delta_M(N) = K$. Suppose that $\delta_M(N) \ne N$. By the definition of $\delta_M(N)$ and Lemma 2.7 (1) it is suffice to show $K \le L$, for any submodule L of L with N/L is M- singular simple . If $L \le N$ such that N/L is M- singular simple, then $K \cap L$ is $\delta_M -$semimaximal submodule of N Hence, by the minimality of K, $K \cap L = K$ and so $K \le L$.

$(c) \Rightarrow (a)$. If $N = \delta_M(N)$, then N is Artinian by Theorem 2.3. Suppose that $N \ne \delta_M(N)$. Then $\delta_M(N) = \bigcap_{i=1}^{n} L_i$, where N/L_i is M- singular simple for each i=1,…n Therefore $N/\delta_M(N)$ is isomorphic to a submodule of the finitely generated semisimple module $\bigoplus_{i=1}^{i=n} N/L_i$. Hence $N/\delta_M(N)$ and so N is Artinian.

(d)⇒(a). Suppose that N is an amply δ_M^- supplemented which satisfies DCC on δ_M^- supplement submodules and δ_M^- small submodules. Then, by Theorem 2.3, $\delta_M(N)$ is Artinian and hence it is suffices to show $N/\delta_M(N)$ is Artinian. $N/\delta_M(N)$ is semisimple by Lemma 1.3.

We claim that $N/\delta_M(N)$ is Noetherian.

Suppose that $\delta_M(N) \le N_1 \le N_2 \le \cdots$ is ascending chain of submodules of N.

We show by induction there exists descending chain of submodules $K_1 \ge K_2 \ge \cdots$ such that K_i is $\delta_M -$supplement N_i of in n for each $i \ge 1$.

Since $N=N_1+N$ and N is amply δ_M^- supplemented, there exists aδ_M^- supplement $K_1$of N_1 in N Then $N=N_1+K_1$. Again since $N=N_2+K_1$,K_1, contains a δ_M^- supplement K_2 of N_2 in N. Now assume $r \ge 1$ and there is a descending $K_1 \ge K_2 \ge \cdots \ge K_r$ of submodules such that K_1 is δ_M^- supplementet of N_I in N for each i=1,2,…r Hence $N = N_r + K_r$ and so $N = N_{r+1} + K_r$. Again since Nis amply δ_M^- supplemented, we have a δ_M^- supplement K_{r+1} of N_{r+1} in N Proceeding in this way we see that there exists a descending chain of submodules $K_1 \ge K_2 \ge \cdots$ such that K_i is $\delta_M -$supplement of N_i in N for each $i \ge 1$. By the hypothesis there exists a positive integer m such that $K_n = K_m$, for each $n \ge m$. Since $N=N_i+K_i$ and $N_i \cap K_i \subseteq \delta_M(N)$, $N/\delta_M(N) = N_i/\delta_M(N) \oplus (K_i + \delta_M(N)/\delta_M(N)$. Thus $N_n = N_m$, for each $n \ge m$. Therefore $N/\delta_M(N)$ is Noetherian and hence finitely generated. Thus $N/\delta_M(N)$ is Artinian.

Note: The condition N is amply δ_M^- supplemented in the statement (d) in Theorem 2.8 cannot be deleted (see the following example).

Example 2.9: Take RZ and M=Z It is clear that $M \in \sigma[M]$,M satisfies DCC on δ_M^- supplement submodules and δ_M^- small submodules, but M is not Artinian.

The next corollary follows from the proof of $(b) \Rightarrow (c)$ in 2.8 and Lemma 2.7(1).

Corollary 2.9: If N satisfies one of the conditions of Theorem 2.8, then $\delta_M(N)$ is the least $\delta_M -$semimaximal submodule of N .

Corollary 2.10: The following statements are equivalent for any R- moduleN .

a) N is Artinian.
b) N satisfies DCC on $\delta_N -$ small submodules and on $\delta_N -$ semimaximal submodules.
c) N satisfies DCC on $\delta_N -$ small submodules and $\delta_N(N)$ is $\delta_N -$semimaximal submodule.
d) N is amply $\delta_N -$supplemented satisfies DCC on $\delta_N -$small submodules and $\delta_N -$ supplement submodules.
e) N satisfies DCC on $\delta -$ small submodules and on $\delta -$ semimaximal submodules.
f) N satisfies DCC on $\delta -$ small submodules and $\delta(N)$ is $\delta_N -$ semimaximal submodule.
g) N is amply $\delta -$supplemented satisfies DCC on $\delta -$small submodules and $\delta -$supplement submodules.

Proof: (a)⇔(b)⇔(c) ⇔(d) is by taking M=N in Theorem 2.8 and (a) ⇔(e) ⇔(f) ⇔(g) by taking M=R in 2.8.

Remark: The equivalence of (a,e,f,g) has been proved by Wang (2007), Proposition 2.8 and Theorem (3.10) Then Theorem 2.8 is an extension of such results.

Corollary 2.12: A finitely generated $\delta_M -$supplemented module N in $\sigma[M]$ is Artinian if and only if N satisfies DCC on $\delta_M -$small submodules.

Proof: The necessary part is trivial.
Sufficiently part, suppose that N is a finitely generated δ_M – supplemented module in σ[M] satisfies DCC on δ_M – small submodules. Then, by Lemma 1.3, $N/\delta_M(N)$ is semisimple and hence it must be Artinian as N is finitely generated. By the hypothesis and 2.3, $\delta_M(N)$ is Artinian. Thus N is Artinian.

We end this Article by showing that every factor module of a δ_M – supplemented module that satisfies ACC on δ_M – small submodules is also satisfies ACC on δ_M – small submodules.

Theorem 2.13: Let $N \in \sigma[M]$ be δ_M- supplemented module. If N satisfies ACC on δ_M- small submodules, then so does every factor modules of N .

Proof. Let $L \le N$ and let $L_1/L \le L_2/L \le \cdots$ be an ascending chain of a δ_M – small submodules of N/L. Since N is a δ_M – supplemented module and $L \le N$, there exists a submodule K of N such that N= L+K and $L \cap K \ll_{\delta_M} K$. Hence $N/L \cong (L+K)/L \cong K/L \cap K$. Let $f: N/L \to K/L \cap K$ be an isomorphism. Therefore for each $i \ge 1$, there exists a submodule K_i of N containing $L \cap K$ such that $f(L_i/L) = K_i/K \cap L$. Hence, by Lemma 1.1, $f(L_i/L) = K_i/K \cap L \ll_{\delta_M} K/L$. Now we show that $K_i \ll_{\delta_M} N$, for each $i \ge 1$. Suppose that $X \le N$ such that $N = K_i + X$, with N/X M- singular. Then $N/K \cap L = K_i/K \cap L + (X + L \cap K)/L \cap K$. But $K_i/K \cap L \ll_{\delta_M} K/L$ and $N/(X + L \cap K)$ is M- singular. So $N/K \cap L = (X + L \cap K)/L \cap K$ and hence $N = (L \cap K) + X$. Therefore N=X. Thus we have a sending chain $K_1 \le K_2 \le \cdots$ of δ_M – small submodules of N. Then, by the hypothesis, there exists a positive integer n such that $K_n = K_{n+1} = \cdots$.

This implies $L/L_n = L/L_{n+1} = \cdots$.Therefore N/L satisfies ACC on δ_M – small submodules.

CONCLUSION

For any module N in σ[M] we have obtained a necessary and sufficient conditions for the sum of all δ_M – small submodules of N to has a finite uniform dimension. Also it is shown that (i) the sum of all δ_M – small submodules of N is Noetherian (Artinian) if and only if N satisfies ACC (DCC) on δ_M – small submodules. (ii) Every factor module of a δ_M – supplemented module in σ[M] with ACC on δ_M – small submodules is also has ACC on δ_M – small submodules . (iii) N is Artinian if and only if N satisfies DCC on δ_M – small submodules and on δ_M – semimaximal submodules if and only if N amply δ_M – supplemented satisfies DCC on δ_M – small submodules and on δ_M – supplement submodules. (iv) If N is finitely generated δ_M – supplemented, then N is Artinian if and N only if N satisfies DCC on δ_M – small submodules.

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