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On the Theory of Intuitionistic Fuzzy N-Inner Product Spaces with Applications

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Abstract: Problem statement: The purpose of this study was to introduce the notion of Intuitionistic fuzzy n-inner product space and to prove a fixed point theorem in complete intuitionistic fuzzy ninner product space. **Conclusion/Recommendations:** Further this result is applied to obtain the existence and uniqueness of solution for linear volterra integral equation.

Key words: Volterra integral equation, intuitionistic fuzzy ninner, complete intuitionistic fuzzy, fixed point theorem, linear space respectively, intuitionistic fuzzy

INTRODUCTION

An interesting theory of 2-inner product space and n-inner product space has been effectively constructed by (Diminnie *et al*., 1977). It was further investigated and developed by (Misiak 1989a). Recent results about n-inner product space can be viewed in (Misiak, 1989b; Cho *et al*., 2002; Cho, 2001). Gunawan and Mashadi 1986) and Malceski (1997), there is a study about the origin and development of n-normed linear space. Fuzzy set theory is useful tool to describe situation in which the data are imprecise or vague. Fuzzy set theory was formalized by Professor (Zadeh, 1965) at the University of California in 1965. After that a lot of studies have been done regarding fuzzy set. The concept of fuzzy set theory may have very important applications in quantum particle physics, medical diagnosis, traffic control system and information retrieval. Different authors introduced the definitions of fuzzy inner product space in (El-Abyad and El-Hamauly, 1991; Cheng and Mordeson, 1994) and fuzzy normed linear space in (Narayanan and Vijayabalaji, 2005; Felbin, 1992; 1993; 1999; Krishna and Sarma, 1994; Bag and Samanta, 2003; Gahler, 1964; Katsaras, 1984; Rhie *et al*., 1997).

Atanassov (1986) introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. There has been much progress in the study of intuitionistic fuzzy sets by many authors. Recently in (Vijayabalaji and Thillaigovindan, 2007; Vijayabalaji *et al*., 2007). Thillaigovindan introduced the notion of fuzzy n-inner product space and the notion of intuitionistic fuzzy n-normed linear space respectively.

In this study, we introduce the notion of Intuitionistic fuzzy n-inner product space as a generalization of fuzzy n-inner product space and prove a fixed point theorem in complete intuitionistic fuzzy n-inner product space. Further this result is applied to obtain the existence and uniqueness of solution for linear volterra integral equation.

Definition 1 (Vijayabalaji *et al*., 2007): Let n be a natural number greater than 1 and X be a real linear space of dimension greater than or equal to n and let (•, • | • , ... , •) be a real valued function on $\underbrace{X\times X\times ...\times X}_{n+1} = X^{n+1}$ Satisfying the following conditions:

- $(x, x|x_2, \ldots, x_n) \geq 0$
- $(x, x|x_2, \ldots, x_n) = 0$ if any only if $x, x_2, \ldots, x_n$ are linearly dependent
- $(x, y|x_2, \ldots, x_n) = (y, x|x_2, \ldots, x_n)$
- $(x, y|x_2, \ldots, x_n)$ is invariant under any permutation of $x_2, \ldots, x_n$
- $(x, x|x_2, \ldots, x_n) = (x_2, x_2|x, x_3, \ldots, x_n)$
- $(ax, x|x_2, \ldots, x_n) = a(x, x|x_2, \ldots, x_n)$ for every $a \in$ R(real)
- $(x + x', y|x_2, \ldots, x_n) = (x, y|x_2, \ldots, x_n) + (x', y|x_2, \ldots, x_n)$

Then (•, • | •, . . . , •) is called an n-inner product on X and (X, (•, • | •, . . . , •)) is called an n-inner product space.

Definition 2 (Vijayabalaji *et al*., 2007): Let $n \in$ N (natural numbers) and X be a real linear space of

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dimension greater than or equal to n. A real valued function $\|\bullet, \ldots, \bullet\|$ on $\underbrace{X \times X \times \ldots \times X}_{n} = X^n$ satisfying the following four properties:

- $\|x_1, x_2, \ldots, x_n\| = 0$ if any only if $x_1, x_2, \ldots, x_n$ are linearly dependent
- $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation
- $\|x_1, x_2, \ldots, ax_n\| = |a|\, \|x_1, x_2, \ldots, x_n\|$, for any $a \in R$ (real)
- $\|x_1, x_2, \ldots, x_{n-1}, y + z\| \le \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$

Is called an n-norm on X and the pair $(X, \|\bullet, \ldots, \bullet\|)$ is called an n-normed linear space.

Remark 3 (Vijayabalaji *et al.*, 2007): If an n-inner product space $(X, (\bullet, \bullet \mid \bullet, \ldots, \bullet))$ is given then $\|x_1, x_2, \ldots, x_n\| = \sqrt{(x_1, x_1 \mid x_2, \ldots, x_n)}$ defines an n-norm on X. Further the following extension of Cauchy-Buniakowski inequality is also true:

$$|(x, y \mid x_2, \ldots, x_n) \le \sqrt{(x, x \mid x_2, \ldots, x_n)}\sqrt{(y, y \mid x_2, \ldots, x_n)}$$

Definition 4: A binary operation Δ: $[0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

- Δ is associative and commutative
- Δ is continuous
- $a \,\Delta\, 1 = a$ for all a Î [0, 1]
- $a \,\Delta\, b \le c \,\Delta\, d$ whenever $a \le c$ and $b \le d$, for each a, b, c, $d \in [0, 1]$

Example 5: Two typical examples of continuous t-norm are $a \,\Delta\, b = ab$ and $a \,\Delta\, b = \min(a, b)$.

Definition 6: A binary operation ∇: $[0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-conorm if it satisfies the following conditions:

- ∇ is associative and commutative
- ∇ is continuous,
- $a \,\nabla\, 0 = a$ for all $a \in [0, 1]$,
- $a \,\nabla\, b \le c \,\nabla\, d$ whenever $a \le c$ and $b \le d$, for each a, b, c, d Î [0, 1]

Example 7: Two typical examples of continuous t-conorm are $a \,\nabla\, b = \min(a+b, 1)$ and $a \,\nabla\, b = \max(a, b)$.

Definition 8 (Atanassov, 1986): Let E be any set. An intuitionistic fuzzy set A of E is an object of the form A $= \{(x\ \mu_A(x)\ \ \gamma_A(x)) \mid x \in E\}$, where the functions: μ_A: E [0,1] and γ_A $E \to [0,1]$ *A* g denote the degree of membership and the non-membership of the element x $\in$ E respectively and for every $x \in E$, $0 \le (x) + \mu_A(x) + \gamma_A(x) \le 1$.

Definition 9 (Vijayabalaji and Thillaigovindan, 2007): An intuitionistic fuzzy n-normed linear space is an object of the form:

$$\left\{\begin{matrix}(X, N(x_1, x_2, \ldots, x_n, t), N' \\ (x_1, x_2, \ldots, x_n, t)) \mid (x_1, x_2, \ldots, x_n) \in X^n\end{matrix}\right\}$$

Where X is a linear space over a field F, Δ is a continuous t-norm, ∇ is a continuous tconorm and N, N' are fuzzy sets on $X^n \times R$ (R-set of real numbers) satisfying following conditions:

- $N(x_1, x_2, \ldots, x_n, t) + N'(x_1, x_2, \ldots, x_n, t) \le 1$
- For all $t \in R$ with $t \le 0$, $N(x_1, x_2, \ldots, x_n, t) = 0$
- For all $t \in R$ with $t > 0$, $N(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent
- $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$
- For all $t \in R$ with $t > 0$, $N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$

$$\text{if } c \ne 0, c \in F(\text{field})$$

- For all $s, t \in R$:

$$N(x_1, x_2, \ldots, x_n + x_n', s + t) \ge \Delta\{N(x_1, x_2, \ldots, x_n, s), N\}(x_1, x_2, \ldots, x_n', t)$$

- $N(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in R$ and:

$$\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1$$

- For all $t \in R$ with $t \le 0$, $N'(x_1, x_2, \ldots, x_n, t) = 1$.
- For all $t \in R$ with $t > 0$, $N'(x_1, x_2, \ldots, x_n, t) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.
- $N'(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.
- For all $t \in R$ with $t > 0$, $N'(x_1, x_2, \ldots, cx_n, t) = N'(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$:

$$\text{if } c \ne 0, c \in F(\text{field})$$

- For all s, t $\in$ R:

$$N'(x_1, x_2, ..., x_n + x_n^{'}, s + t) \le \nabla$$
$$\{N'(x_1, x_2, ..., x_n, s), N'(x_1, x_2, .., x_n, t)\}$$

N' (x_1, x_2, . . . , x_n, t) is a non-increasing function of t $\in$ R and:

$$\lim_{t\to\infty} N'(x_1, x_2, ..., x_n, t) = 0$$

Definition 10: Let X be a linear space over a field F. A fuzzy subset J: $X^{n+1} \times$ R (Real numbers) is called a fuzzy n- inner poduct on X if and only if:

- For all t $\in$ R with t ≤ 0, J(x, x|x_1, . . . , x_{n-1}, t) = 0
- For all t $\in$ R with t > 0, J(x, x|x_1, . . . , x_{n-1}, t) = 1 if and only if x, x_1, . . . , x_{n-1} are linearly dependent
- For all t > 0, J (x, y|x_1, . . . , x_{n-1}, t) = J (y, x|x_1, . . . , x_{n-1}, t)
- J(x, y|x_1, . . . , x_{n-1}, t) is invariant under any permutation of x_1, . . . , x_{n-1}
- For all t > 0, J(x, x|x_1, . . . , x_{n-1}, t) = J(x_1, x_1|x, x_2, . . . , x_{n-1} , t)
- For all t > 0, J(ax, bx|x_1, . . . , x_{n-1}, t) = J(x, x|x_1, . . . , x_{n-1}, $\frac{t}{|ab|}$), a, b$\in$R(real)
- For all s, t$\in$R:

$$J(x, y | x_1, ..., x_{n-1}, \sqrt{ts}) \ge \min$$
$$\{J(x | x_1.t), J(y, yZ(x_1, ..., x_{n-1}, t)\}$$

J(x + *x*¢ , y|x_1 , . . . , x_{n-1}, t + s) _min{J(x, y|x_1, . . . , x_{n-1}, t), J(*x*¢ , y|x_1, . . . , x_{n-1}, s)}.

- For all s, t $\in$ R with s > 0, t > 0:

$$J(x, y | x_1, ..., x_{n-1}, \sqrt{ts}) \ge \min\{J(x, x | x_1, ..., x_{n-1}, t),$$
$$J(y, y | x_1, ..., x_{n-1}, t)\}$$

- J(x, y|x_1, . . . , x_{n-1}, t) is a non-decreasing function of t$\in$R and:

$$\lim_{t\to\infty} J(x, y | x_1, ..., x_{n-1}, t) = 1$$

Then (X, J) is called a fuzzy n-inner product space or in short f-n-IPS.

Intuitionistic fuzzy n-inner product space: We introduce the notion of intuitionistic fuzzy n-inner product space as a generalization of Definition 10 as follows.

Definition 11: An intuitionistic fuzzy n-inner poduct space is an object of the form:

$$A = \left\{ \begin{matrix} (X, J(x, y | x_1, x_2, ..., x_{n-1}, t) \\ , J'(x, y | x_1, x_2, ..., x_{n-1}, t)) | (x, y, x_1, x_2, ..., x_{n-1}) \in X^{n+1} \end{matrix} \right\}$$

where, X is a linear space over a field F, Δ is a continuous t-norm, ∇ is a continuous tconorm and J, J ' are fuzzy sets on X^{n+1} × R (R-set of real numbers) satisfying following conditions:

- J(x, y|x_1, . . . , x_{n-1}, t) + J ' (x, y|x_1, . . . , x_{n-1}, t) ≤ 1
- For all t $\in$ R with t ≤ 0, J(x, x|x_1, . . . , x_{n-1}, t) = 0 (or θ, the null vector)
- For all t$\in$R with t > 0, J(x, x|x_1, . . . , x_{n-1}, t) = 1 if and only if x, x_1, . . . , x_{n-1} are linearly dependent
- For all t > 0, J(x, y|x_1, . . . , x_{n-1}, t) = J(y, x|x_1, . . . , x_{n-1}, t)
- J(x, y|x_1, . . . , x_{n-1}, t) is invariant under any permutation of x_1, . . . , x_{n-1}
- For all t > 0, J(x, x|x_1, . . . , x_{n-1}, t) = J(x_1, x_1 |x, x_2, . . . , x_{n-1}, t)
- For all t > 0, J(ax, bx|x_1, . . . , x_{n-1}, t) = J(x, x|x_1, . . . , x_{n-1}, $\frac{t}{|ab|}$) a, b$\in$R(real)
- For all s, t $\in$ R:

$$J(x + x', y | x_1, ..., x_{n-1}, t + s) \ge \Delta$$
$$\{J(x, y | x_1, ..., x_{n-1}, t)(x_1, ..., x_{n-1}, s)\}$$

- For all s, t $\in$ R with s > 0, t > 0:

$$J(x, y | x_1, ..., x_{n-1}, \sqrt{ts}) \ge \Delta\{J(x, x | x_1, ..., x_{n-1}, t),$$
$$J(y, y | x_1, ..., x_{n-1}, t)\}$$

- J(x, y|x_1, . . . , x_{n-1}, t) is a non-decreasing function of t$\in$R and:

$$\lim_{t\to\infty} J(x, y | x_1, ..., x_{n-1}, t) = 1.$$

- For all t $\in$ R with t ≤ 0, J ' (x, x|x_1, . . . , x_{n-1}, t) = 1
- For all t $\in$ R with t > 0, J ' (x, x|x_1, . . . , x_{n-1}, t) = 0 if and only if x, x_1, . . . , x_{n-1} are linearly dependent
- For all t > 0, J' (x, y|x_1, . . . , x_{n-1}, t) = J' (y, x|x_1, . . . , x_{n-1}, t)
- J' (x, y|x_1, . . . , x_{n-1}, t) is invariant under any permutation of x_1, . . . , x_{n-1}
- For all t > 0, J' (x, x|x_1, . . . , x_{n-1}, t) = J' (x_1 , x_1 |x, x_2, . . . , x_{n-1}, t)

- For all t > 0, J' (ax, bx|x_1, . . . , x_{n-1}, t) = J ' (x, x|x_1, . . . , x_{n-1}, $\frac{t}{|ab|}$) a, b∈ R
- For all s, t ∈ R:

$$J'(x+x',y|x_1,...,x_{n-1},t+s)\le\nabla\{x,y|x_1,...,x_{n-1},t\},$$
$$J'(x',y|x_1,...,x_{n-1},s)\}$$

- For all s, t ∈ R with s > 0, t > 0:

$$J'(x,y|x_1,...,x_{n-1},\sqrt{ts})\le\nabla\{J'(x,x|x_1,....,x_{n-1},t),$$
$$J'(x',y|x_1,..,x_{n-1},s)\}$$

J' (x, y|x_1, . . . , x_{n-1}, t) is a non-increasing function of t∈R and:

$$\lim_{t\to\infty}J'(x,y|x_1,..,x_{n-1},t)=0$$

Then (X, J, *J* ' ,D ,Ñ) is called an intuitionistic fuzzy n-inner product space or in short IF-n-IPS.

Example 12: Let (X, (•, • | •, . . . , •)) be an n- inner product space. Define a ∇ b = min (a, b) and a ∈ b = max(a, b), for all a, b ∈ [0, 1]:

$$J\left(x,y|x_1,...,x_{n-1},t\right)=\begin{cases}\dfrac{t}{t+|\left(x,y|x_1,...,x_{n-1}\right)} \\ \text{when } t>0, t\in R, \left(x,y|x_1,...,x_{n-1}\right)\in X^{n+1} \\ 0, \text{ when } t|\le 0\end{cases}$$

And:

$$J'\left(x,y|x_1,...,x_{n-1},t\right)=\begin{cases}\dfrac{\left|\left(x,y|x_1,...,x_{n-1}\right)\right|}{t+|\left(x,y|x_1,...,x_{n-1}\right)} \\ \text{when } t>0, t\in R, \left(x,y|x_1,...,x_{n-1}\right)\in X^{n+1} \\ 1, \text{ when } t|\le 0\end{cases}$$

Then (X, J, J', Δ,∇) is an IF-n-IPS.

Proof:

- Clearly $J\left(x,y|x_1,..,x_{n-1},t\right)+J'\left(x,y|x_1,...,x_{n-1},t\right)\le 1$
- For all t∈R with t ≤ 0 we have by our definition, J (x, x|x_1,…, x_{n-1}, t) = 0
- For all t ∈ R with t > 0we have, J (x, x|x_1,.., x_{n-1},t) = 1:

$$\frac{t}{t+\left(x,x|x_1,..,x_{n-1}\right)}=1\Leftrightarrow\left|\left(x,x|x_1,...,x_{n-1}\right)\right|=0$$

|(x,x|x_1,…,x_{n-1})|=0 ⇔ x,|x_1,..,x_{n-1} are linearly dependent.

- For all t > 0:

$$J\left(x,y|x_1,...,x_{n-1},t\right)=\frac{t}{t+\left|\left(x,y|x_1,...,x_{n-1}\right)\right|}$$
$$=\frac{t}{t+\left|\left(y,x|x_1,...,x_{n-1}\right)\right|}=J(y,x|x_1,...,x_{n-1},t)$$

As (x, x|x_1. . . x_{n-1}, t) is invariant under any permutation of x_1 , . . . , x_{n-1}.

We have J(x, y|x_1. . . x_{n-1}, t) is invariant under any permutation of x_1, . . . , x_{n-1}.

For all t > 0:

$$J\left(x,x|x_1,...,x_{n-1},t\right)=\frac{t}{t+|\left(x_1,x_1|x,...,x_{n-1}\right)|}$$
$$=\frac{t}{t+|\left(x_1,x_1|x,...,x_{n-1}\right)|}=J\left(x_1,x_1|x,...,x_{n-1},t\right)$$

For all t > 0:

$$J\left(x,x|x_1,...,x_{n-1},\frac{t}{|ab|}\right)=\frac{\frac{t}{|ab|}}{\frac{t}{|ab|}+|\left(x,x|x_1,...,x_{n-1}\right)|}$$
$$=\frac{t}{|t+|ab||\left(x,x|x_1,...,x_{n-1}\right)|}$$
$$=\frac{t}{t+|\left(ax,bx|x_1,....x_{n-1}\right)|}$$
$$=J\left(ax,bx|x_1,...,x_{n-1},t\right)$$

If (a) s + t < 0 (b) s = t = 0 (c) s + t > 0; s > 0, t < 0; s < 0, t > 0,Then the above relation is obvious. If (d) s > 0, t > 0, s + t > 0. Then Without loss of generality assume that:

$$J\left(x,y|x_1,...,x_{n-1},t\right)\le J\left(x',y|x_1,...,x_{n-1},s\right)$$
$$\frac{t}{t+|\left(x,y|x_1,...,x_{n-1}\right)|}\le\frac{s}{s+|\left(x,y|x_1,...,x_{n-1}\right)|}$$
$$\frac{t+|\left(x,y|x_1,...,x_{n-1}\right)|}{t}\ge\frac{s+|\left(x',y|x_1,...,x_{n-1}\right)|}{s}$$
$$1+\frac{|\left(x,y|x_1,...,x_{n-1}\right)|}{t}\ge 1+\frac{|\left(x',y||x_1,...,x_{n-1}\right)|}{s}$$
$$\frac{|\left(x,y|x_1,...,x_{n-1}\right)|}{t}\ge\frac{|\left(x',y|x_1,...,x_{n-1}\right)|}{s}$$
$$\frac{|\left(x,y|x_1,...,x_{n-1}\right)}{t}\ge|\left(x',y|x_1,....,x_{n-1}\right)|$$

$$|(x,y|x_1,\ldots,x_{n-1})|+\frac{|(x,y|x_1,\ldots,x_{n-1})}{t}\geq$$
$$|(x,y|x_1,\ldots,x_{n-1})|+|(x',y|x_1,\ldots,x_{n-1})|$$
$$\left(1+\frac{s}{t}\right)Z(x,y|x_1,\ldots,x_{n-1})|\geq|(x+x',y|x_1,\ldots,x_{n-1})|$$
$$\frac{s+t}{t}|(x,y|x_1,\ldots,x_{n-1})|\geq|(x+x',y|x_1,\ldots,x_{n-1})|$$
$$\frac{|(x,y|x_1,\ldots,x_{n-1})}{t}\geq\frac{|(x+x',y|x_1,\ldots,x_{n-1})|}{s+t}$$
$$1+\frac{|(x,y|x_1,\ldots,x_{n-1})|}{t}\geq 1+\frac{|(x+x',y|x_1,\ldots,x_{n-1})|}{s+t}$$
$$\frac{t+|(x,y|x_1,\ldots,x_{n-1})}{t}\geq\frac{s+t+|(x+x',y|x_1,\ldots,x_{n-1})|}{s+t}$$
$$\frac{t}{t+|(x,y|x_1,\ldots,x_{n-1})|}\leq\frac{s+t}{s+t+|(x,y|x_1,\ldots,x_{n-1})|}$$
$$\min\{J(x,y|x_1,\ldots,x_{n-1},t),J(x'y|x_1,\ldots,x_{n-1},s)\}$$
$$\leq J(x+x',y|x_1,\ldots,x_{n-1},s+t)$$

Without loss of generality assume that:

J $(x,x|x_1,\ldots,x_{n-1},t)\leq J(y,y|x_1,\ldots,x_{n-1},s)$ for all s, t∈ with s> 0,t>0:

$$J(x,x|x_1,\ldots,x_{n-1})\leq J(y,y|x_1,\ldots,x_{n-1})$$
$$\frac{t}{t+|(x,x|x_1,\ldots,x_{n-1})|}\leq\frac{s}{s+|(y,y|x_1,\ldots,x_{n-1})|}$$
$$\frac{t+|(x,x|x_1,\ldots,x_{n-1})|}{t}\geq\frac{s+|(y,y|x_1,\ldots,x_{n-1})|}{s}$$
$$1+\frac{|(x,x|x_1,\ldots,x_{n-1})|}{t}\geq 1+\frac{|(y,y|x_1,\ldots,x_{n-1})|}{s}$$
$$|(x,x|x_1,\ldots,x_{n-1})|.\frac{s|(x,x|x_1,\ldots,x_{n-1})|}{t}$$
$$\geq|(x,x|x_1,\ldots,x_{n-1})||(y,y|x_1,\ldots,x_{n-1})|$$
$$\frac{|(x,x|x_1,\ldots,x_{n-1})|}{t}\geq\frac{|(y,y|x_1,\ldots,x_{n-1})|}{t}$$
$$\frac{s|(x,x|x_1,\ldots,x_{n-1})|}{t}\geq|(y,y|x_1,\ldots,x_{n-1})|$$

By Remark:

$$\frac{s|(x,x|x_1,\ldots,x_{n-1})|^2}{t}\geq|(x,y|x_1,\ldots,x_{n-1})|^2$$
$$\frac{|(x,x|x_1,\ldots,x_{n-1})|^2 s}{t^2}\geq|(x,y|x_1,\ldots,x_{n-1})|^2$$
$$\frac{|(x,x|x_1,\ldots,x_{n-1})|^2}{t^2}\geq\frac{|(x,y|x_1,\ldots,x_{n-1})|^2}{st}$$

Taking square root on both sides:

$$\frac{|(x,x|x_1,\ldots,x_{n-1})|}{t}\geq\frac{|(x,y|x_1,\ldots,x_{n-1})|}{\sqrt{st}}$$
$$1+\frac{|(x,x|x_1,\ldots,x_{n-1})|}{t}\geq 1+\frac{|(x,y|x_1,\ldots,x_{n-1})|}{\sqrt{st}}$$
$$\frac{|(x,x|x_1,\ldots,x_{n-1})|}{t}\geq\frac{\sqrt{st}+|(x,y|x_1,\ldots,x_{n-1})|}{\sqrt{st}}$$
$$\frac{t}{t+|(x,x|x_1,\ldots,x_{n-1})|}\leq\frac{\sqrt{st}}{\sqrt{st}+|(x,y|x_1,\ldots,x_{n-1})|}$$
$$\frac{t}{t+|(x,x|x_1,\ldots,x_{n-1},t)}\leq\frac{\sqrt{st}}{\sqrt{st}+|(x,y|x_1,\ldots,x_{n-1})|}$$
$$\min\{J(x,x|x_1,\ldots,x_{n-1},t),J(y,y|x_1,\ldots,x_{n-1},s)\}$$
$$\leq J(x,y|x_1,\ldots,x_{n-1},\sqrt{ts})$$

For all $t_1 < t_2 \leq 0$ then, by our definition:

$$J(x,y|x_1,\ldots,x_{n-1})|=J(x,y|x_1,\ldots,x_{n-1})|=0.$$

Suppose $t_2 > t_1 > 0$ then:

$$\frac{t_2}{t_2+|(x,y|x_1,\ldots,x_{n-1})|}-\frac{t_1}{t_1+|(x,y|x_1,\ldots,x_{n-1})|}$$
$$=\frac{|(x,y|x_1,\ldots,x_{n-1})|(t_2-t_1)}{(t_2+|(x,y|x_1,\ldots,x_{n-1})|)(t_1+|(x,y|x_1,\ldots,x_{n-1})|)}$$
$$\geq 0,\text{for all}(x,y|x_1,\ldots,x_{n-1})\in X^{n+1}$$
$$\frac{t_2}{t_2+|(x,y|x_1,\ldots,x_{n-1})|}\geq\frac{t_1}{t_1+|(x,y|x_1,\ldots,x_{n-1})|}$$
$$J(x,y|x_1,\ldots,x_{n-1}\ t_2)\geq J(x,y|x_1,\ldots,x_{n-1},t_1)$$

Thus J $(x\ y|x_1,\ldots,\ x_{n-1},\ t)$ is non-decreasing function. Also:

$$\lim_{t\to\infty}J(x,y|x_1,\ldots,x_{n-1})=\lim_{t\to\infty}\frac{t}{t+|(x,y|x_1,\ldots,x_{n-1})|}$$
$$=\lim_{t\to\infty}\frac{t}{t\left(1+\frac{1}{t}|x,y|x_1,\ldots,x_{n-1}\right)|}=1.$$

For all t ∈ R with t ≤ 0 we have by our definition:

$$J'(x,x|x_1,\ldots,x_{n-1})=1.$$

For all t ∈ R with *t* > 0we have:

$$J'(x,x|x_1,\ldots,x_{n-1},t)=0$$
$$\frac{|(x,x|x_1,\ldots,x_{n-1})|}{t+|(x,x|x_1,\ldots,x_{n-1})|}=0|(x,x|x_1,\ldots,x_{n-1})|=0$$
$$|(x,x|x_1,\ldots,x_{n-1})|=0x.x_1,\ldots,x_{n-1}$$
$$\text{are linearly dependent}$$

For all t > 0:

$$J'(x,y|x_1,...,x_{n-1},t) = \frac{|(x,y|x_1,...,x_{n-1})|}{t+|(x,y|x_1,...,x_{n-1})|}$$
$$= \frac{|(y,x|x_1,...,x_{n-1})|}{t+|(y,x|x_1,...,x_{n-1})|}$$
$$= J'(y,x|x_1,...,x_{n-1},t).$$

As (x, x|x_1, . . . , x_{n-1}, t) is invariant under any permutation of x_1, . . . , x_{n-1}.
We have J' (x, y|x_1, . . . , x_{n-1}, t) is invariant under any permutation of x_1, . . . , x_{n-1}.
For all t > 0:

$$J'(x,x|x_1,...,x_{n-1}) = \frac{\left|(x,x|x_1,...,x_{n-1})\right|}{t+\left|(x,x|x_1,...,x_{n-1})\right|}$$
$$= \frac{\left|(x_1.x_1|x,...,x_{n-1})\right|}{t+\left|(x_1,x_1|x,...,x_{n-1})\right|}$$
$$= J'(x_1,x_1|x,...,x_{n-1},t)$$

$$J'\left(x,x|x_1,...,x_{n-1},\frac{t}{|ab|}\right) = \frac{|(x,x|x_1,...,x_{n-1})|}{\frac{t}{|ab|}+|(x,x|x_1,...,x_{n-1})}$$
$$= \frac{|ab||(x,x|x_1,...,x_{n-1})|}{t+|ab||(x,x|x_1,...,x_{n-1})|}$$
$$= \frac{\left|(ax,bx|x_1,...,x_{n-1})\right|}{t+|(ax,bx|x_1,...,x_{n-1})|}$$
$$= J'(ax,bx|x_1,...,x_{n-1},t)$$

If (a) s + t < 0 (b) s = t = 0 (c) s + t > 0; s > 0, t < 0; s < 0, t > 0,

Then the above relation is obvious. If (d) s > 0, t > 0, s + t > 0. Then Without loss of generality assume that:

$$J'(x,y|x_1,...,x_{n-1}) \le J'(x',y|x_1,...,x_{n-1})$$
$$\frac{\left|(x,y|x_1,...,x_{n-1})\right|}{t+\left|(x,y|x_1,...,x_{n-1})\right|} \le \frac{\left|(x',y|x_1,...,x_{n-1})\right|}{s+\left|(x',y|x_1,...,x_{n-1})\right|}$$
$$\frac{t+\left|(x,y|x_1,...,x_{n-1})\right|}{\left|(x,y|x_1,...,x_{n-1})\right|} \ge \frac{s+\left|(x',y|x_1,...,x_{n-1})\right|}{\left|(x',y|x_1,...,x_{n-1})\right|}$$
$$1+\frac{t}{\left|(x,y|x_1,...,x_{n-1})\right|} \ge 1+\frac{s}{\left|(x',y|x_1,...,x_{n-1})\right|}$$
$$\frac{t}{\left|(x,y|x_1,...,x_{n-1})\right|} \ge \frac{s}{\left|(x',y|x_1,...,x_{n-1})\right|}$$
$$\frac{t\left|(x',y|x_1,...,x_{n-1})\right|}{s} \ge \left|(x,y|x_1,...,x_{n-1})\right|$$

$$\left|(x',y|x_1,...,x_{n-1})\right| + \frac{t\left|(x',y|x_1,...,x_{n-1})\right|}{s} \ge$$
$$\left|(x',y|x_1,...,x_{n-1})\right| + \left|(x,y|x_1,...,x_{n-1})\right|$$
$$\left(1+\frac{t}{s}\right)\left|(x',y|x_1,...,x_{n-1})\right| \ge \left|(x+x',y|x_1,...,x_{n-1})\right|$$
$$\left(\frac{s+t}{s}\right)\left|(x',y|x_1,...,x_{n-1})\right| \ge \left|(x+x',y|x_1,...,x_{n-1})\right|$$
$$\frac{\left|(x',y|x_1,...,x_{n-1})\right|}{s} \ge \frac{\left|(x+x',y|x_1,....,x_{n-1})\right|}{s+t}$$
$$\frac{s}{\left|(x',y|x_1,...,x_{n-1})\right|} \le \frac{s+t}{\left|(x+x',y|x_1,...,x_{n-1})\right|}$$
$$1+\frac{s}{\left|(x',y|x_1,...,x_{n-1})\right|} \le 1+\frac{s+t}{\left|(x+y|x_1,...,x_{n-1})\right|}$$
$$\frac{s+\left|(x',y|x_1,...,x_{n-1})\right|}{\left|(x',y|x_1,...,x_{n-1})\right|} \le \frac{s+t\left|(x+x',y|x_1,...,x_{n-1})\right|}{\left|(x+x',y|x_1,...,x_{n-1})\right|}$$
$$\frac{\left|(x',y|x_1,...,x_{n-1})\right|}{s+\left|(x',y|x_1,....,x_{n-1})\right|} \ge \frac{\left|(x+x',y|x_1,...,x_{n-1})\right|}{s+t+\left|(x+x',y|x_1,...,x_{n-1})\right|}$$
$$\max\{J'(x,y|x_1,...,x_{n-1},t)\},J(x',y|x_1,...,x_{n-1},s) \ge$$
$$J'(x+x',y|x_1,....,x_{n-1},s+t)$$

Without loss of generality assume that J'(x x|x_1,…, x_{n-1}, t) ≤ J' (y,y |x_1,…,x_{n-1},s)for all s, t∈R with s > 0, t > 0:

$$\frac{\left|(x,x|x_1,...,x_{n-1})\right|}{t+\left|(x,x|x_1,....,x_{n-1})\right|} \le \frac{\left|(y,y|x_2,...,x_n)\right|}{s+\left|(y,y|x_1,...,x_{n-1})\right|}$$
$$\frac{t+\left|(x,x|x_1,...,x_{n-1})\right|}{\left|(x,x|x_1,....,x_{n-1})\right|} \ge \frac{s+\left|(y,y|x_2,...,x_n)\right|}{\left|(y,y|x_1,...,x_{n-1})\right|}$$
$$1+\frac{t}{\left|(x,x|x_1,...,x_{n-1})\right|} \ge 1+\frac{s}{\left|(y,y|x_1,...,x_{n-1})\right|}$$
$$\frac{t}{\left|(x,x|x_1,...,x_{n-1})\right|} \ge \frac{s}{\left|(y,y|x_1,...,x_{n-1})\right|}$$
$$\left|(y,y|x_1,...,x_{n-1})\right| \ge \left|(x,x|x_1,...,x_{n-1})\right|.\frac{s}{t}$$
$$\left|(y,y|x_1,...,x_{n-1})\right|.\left|(y,y|x_1,....,x_{n-1})\right|$$
$$\ge \left|(x,x|x_1,...,x_{n-1})\right|\left|(y,y|x_1,....,x_{n-1})\right|.\frac{s}{t}$$

By Remark:

$$\frac{s\left|(x,y|x_1,...,x_{n-1})\right|^2}{t} \le \left|(y,y|x_1,...,x_{n-1})\right|^2$$
$$\frac{\left|(x,y|x_1,...,x_{n-1})\right|^2.s}{st} \le \frac{\left|(y,y|x_1,...,x_{n-1})\right|^2}{s}$$

$$\frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|^2}{st} \le \frac{\left|\left(y,y\,|\,x_1,...,x_{n-1}\right)\right|^2}{s^2}$$

Taking square root on both sides:

$$\frac{\left|\left(x,x\,|\,x_1,...,x_{n-1}\right)\right|}{\sqrt{st}} \le \frac{\left|\left(y,y\,|\,x_1,...,x_{n-1}\right)\right|}{s}$$

$$\frac{s}{\left|\left(y,y\,|\,x_1,...,x_{n-1}\right)\right|} \le \frac{\sqrt{st}}{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}$$

$$1+\frac{s}{\left|\left(y,y\,|\,x_1,...,x_{n-1}\right)\right|} \le 1+\frac{\sqrt{st}}{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}$$

$$\frac{\left|\left(y,y\,|\,x_1,...,x_{n-1}\right)\right|+s}{\left|\left(y,y\,|\,x\ 1,...,x_{n-1}\right)\right|} \le \frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|+\sqrt{st}}{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}$$

$$\frac{\left|\left(y,y\,|\,x_1,...,x_{n-1}\right)\right|}{s+\left|\left(y,y\,|\,x_1,...,x_{n-1}\right)\right|} \ge \frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}{\sqrt{st}+\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}$$

$$\max\left\{J'\left(x,x,|\,x\ 1,...,x_{n-1}\right),J'\left(y,y\,|\,x\ 1,...,x_{n-1}\right)\right\}$$
$$\ge J'\left(x,y\,|\,x_1x_{n-1},\sqrt{ts}\right)$$

For all t1 < $t_2 \le 0$ then, by our definition:

$$J'(x,y\,|\,x_1,...,x_{n-1},t_1) = J'(x,y\,|\,x_1,...,x_{n-1},t_2) = 1$$

Suppose $t_2 > t_1 > 0$ then:

$$\frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}{t_1+\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|} - \frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}{t_2+\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}$$
$$= \frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|\left(t_2-t_1\right)}{\left(t_2+\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|\right)\left(t_1+\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|\right)}$$
$$\ge 0, \text{for all}\,(x,y\,|\,x_1,...,x_{n-1}) \in X^{n+1}$$

$$\frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}{t+\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|} \ge \frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}{t_2+\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}$$

$$J'\left(x,y\,|\,x_1,...,x_{n-1},t_1\right) \ge J'\left(x,y\,|\,x_1,...,x_{n-1},t_2\right)$$

Thus J’ (x y| $x_1,\ldots, x_{n-1}$, t) is non-increasing function. Also:

$$\lim_{t\to\infty} J'\left(x,y\,|\,x_1,...,x_{n-1},t\right) = \lim_{t\to\infty}\frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}{t+\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}$$

$$\lim_{t\to\infty} J'\frac{\left|\left(x,y\,|\,x_1,...,x_{n-1}\right)\right|}{t\left(1+\frac{1}{t}\left|\left(x,y\,|\,x_2,...,x_{n-1}\right)\right|\right)} = 0.$$

Definition 13: We call that t-norm Δ and t-conorm ∇ are h-type if the family of functions $\left\{\Delta^m(t)\right\}_{m=1}^{\infty}$ and $\left\{\nabla^m(t)\right\}_{m=1}^{\infty}$ are equi-continuous at t =1 and t = 0 respectively.

Where:

$$\Delta^1(t) = \Delta(t,t), \Delta^m(t) = \Delta(t,\Delta^{m-1}(t)) \text{and}$$
$$\nabla^1(t) = \nabla(t,t), \nabla^m(t) = \nabla(t,\nabla^{m-1}(t))$$

$t \in [0,1], m = 2, 3,\ldots$

Definition 14: Let (X, J, J’, Δ , ∇) be a IF-n-IP-space.
(i) A sequence $x_n \subset X$ is said to converge to $x \in X$ if $\forall \varepsilon>0$, $\forall a \in (0, 1]$, ΞN, when
$n \ge N$, J(x_n-x, x_n-x|$x_1,\ldots, x_{n-1}$ ε) > 1- a and J’ (x_n-x, x_n-x |$x_1,\ldots, x_{n-1}$ ε)≤ a.
(ii) A sequence $x_n \subset X$ is called a Cauchy sequence if $\forall \varepsilon > 0$, $\forall a \in (0, 1]$, ΞN, when
m, n≥N, J(x_n-x_m, x_n-x_m| $x_1,\ldots, x_{n-1}\varepsilon$) > 1- a and:

$$J'(x_n - x_m, x_n - x_m\,|\,x_1,....,x_{n-1},\varepsilon) \le a$$

CONCLUSION

Theorem 15: Let (X, J, J’, Δ, ∇) be a complete IF-n-IP-space and Δ, ∇ be t-norm and t-conorm of h-type respectively. Let T: (X, J, J’, Δ, ∇) $\to$ (X, J, J', Δ, ∇) be a linear mapping satisfying the following condition:

$$J(Tx,y\,|\,x_1,..,x_{n-1},t) \ge J(x,y\,|\,x_1,..,x_{n-1}\,|,\frac{t}{k(\alpha,\beta)})$$

And:

$$J(Tx,y\,|\,x_1,..,x_{n-1},t) \le J'(x,y\,|\,x_1,..,x_{n-1},\frac{t}{k(\alpha,\beta)})$$

For all x, y, $x_1, x_2,\ldots,x_{n-1} \in X$ $t\ge 0$ $\alpha, \beta \in (0,\infty)$ and k (α, β): $(0,+\infty) \times (0,+\infty) \to (0, 1)$ is a function. Then T has exactly one fixed point $x_* \in X$. Furthermore, for any $x_0 \in X$, the iterative sequence $\{T^n x_0\}$τ-converges to x_*.

Proof: Firstly, we prove that for any $x_0 \in X$, the sequence $\left\{x_m\right\}_{m=0}^{\infty}$ is at τ Cauchy sequence, where $\left\{x_m\right\}_{m=0}^{\infty} = \left\{x_0, x_1 = Tx_0,.., x_m = T^m x_0,..\right\}$

By (8) and (17) of Definition 11, we have:

$$J\left(x_0 - T^m x_0, y\,|\,x_1,....,x_{n-1},\frac{t}{k(\alpha,\beta)}\right) =$$
$$J\left(x_0 - Tx_0 + Tx_0 - T^m x_0, y\,|\,x_1,...,x_{n-1},\frac{t}{k(\alpha,\beta)}\right)$$

$$\geq \Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right)$$

$$J\left(Tx_0 - T^m x_0, y \mid x_1, ..., x_{n-1}, \frac{tk(a,\beta)}{k(a,\beta)}\right)\Bigg)$$

$$\geq \Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$\left.J\left(x_0 - T^{m-1}x_0, y \mid x_1, ..., x_{n-1}, \frac{t}{k(a,\phi)}\right)\right)$$

$$= \Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$\left.J\left(x_0 - Tx_0 + Tx_0 - T^{m-1}x_0, y \mid x_1, ..., x_{n-1}, \frac{t}{k(a,\beta)}\right)\right)$$

$$\geq \Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right),\right.$$

$$\Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$\left.\left.J\left(Tx_0 - T^{m-1}x_0, y \mid x_1, ..., x_{n-1}, \frac{tk(a,\beta)}{k(a,\beta)}\right)\right)\right)$$

$$\geq \Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$,\Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$\left.\left.J,\left(x_0 - T^{m-2}x_0, y \mid x_1, ..., x_{n-1}, \frac{t}{k(a,\beta)}\right)\right)\right)$$

$$\geq .... \geq \Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_n, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$\left.,\Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\phi)}\right)\right)\right.$$

And:

$$J'\left(x_0 - T^m x_0, y \mid x_1, ..., x_{n-1}, \frac{t}{k(\alpha,\beta)}\right) =$$

$$J'\left(x_0 - Tx_0 + Tx_0 - T^m x_0, y \mid x_1, ..., x_{n-1}, \frac{t}{k(\alpha,\beta)}\right)$$

$$\leq \left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$\left.J'\left(Tx_0 - T^m x_0, y \mid x_1, ..., x_{n-1}, \frac{tk(a,\beta)}{k(a,\beta)}\right)\right)$$

$$\leq \nabla\left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right)$$

$$J'\left(x_0 - T^{m-1}x_0, y \mid x_1, ..., x_{n-1} \frac{t}{k(\alpha,\beta)}\right)\Bigg)$$

$$= \nabla\left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right)$$

$$J'\left(x_0 - Tx_0 + T - T^{m-1}x_0, y \mid x_1, ..., x_{n-1} \frac{t}{k(\alpha,\beta)}\right)\Bigg)$$

$$\leq \nabla\left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$,\nabla\left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$\left.\left.J'\left(Tx_0 - T^{m-1}x_0, y \mid x_1, ..., x_{n-1}, \frac{tk(a,\beta)}{k(a,\beta)}\right)\right)\right)$$

$$\leq \nabla\left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$,\nabla\left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right),\right.$$

$$J'\left(x_0 - T^{m-2}x_0, y \mid x_1, ..., x_{n-1}, \frac{t}{k(a,\beta)}\right)$$

$$\leq .... \leq \nabla\left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right),\right.$$

$$\nabla\left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.$$

$$\nabla\left(...,\nabla\left(J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(a,\beta))}{k(a,\beta)}\right)\right.\right.$$

$$,J'\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t}{k(a,\beta)}\right)$$

Because of $k(\alpha, \beta) \in (0, 1)$, therefore we get:

$$\frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)} \leq \frac{t}{k(\alpha,\beta)}$$

As J, J' are non-decreasing and non-increasing respectively, Therefore, we have:

By the property of t-norm and t-conorm, we obtain:

$$J\left(x_0 - T^m x_0, y \mid x_1, ..., x_{n-1}, \frac{t}{k(\alpha,\beta)}\right)$$

$$\geq \Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1} \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right.$$

$$,\Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right.$$

$$\Delta\left(...,\Delta\left(J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right.\right.$$

$$\left.,J\left(x_0 - Tx_0, y \mid x_1, ..., x_{n-1}, \frac{t}{k(\alpha,\beta)}\right)\right)....\Bigg)$$

$$= \Delta^{m-1}\left(J\left(x_0 - Tx_0, y \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right)$$

$$J'\left(x_0 - T^m x_0, y \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right)$$

$$\leq \left(J'\left(x_0 - Tx_0, y \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right.$$

$$,\nabla\left(J'\left(x_0 - Tx_0, y \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right.$$

$$\nabla\left(...,\nabla\left(J'\left(x_0 - Tx_0, y \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right)\right), J'$$

$$\left(x_0, -Tx_0, y \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)$$

$$= \nabla^{m-1}\left(J'\left(x_0 - Tx_0, y \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right)\right)$$

So, for any positive integer m, n, we have:

$$J\left(T^n x_0 - T^{m+n}x_0, T^n x_0 - T^{m+n}x_0 \mid x_1,..,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right) \geq$$

$$J\left(x_0 - T^m x_0 - T^{m+n}x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^{n+1}(\alpha,\beta)}\right)$$

$$\geq J\left(x_0 - T^m x_0, x_0 T^m x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^{2n+1}(\alpha,\beta)}\right)$$

$$\geq \Delta^{m-1}\left(J\left(x_0 - Tx_0, x_0 - T^m x_0 \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k^{2n-1}(\alpha,\beta)}\right)\right)$$

$$\geq \Delta^{m-1}\left(\Delta^{m-1}\left(J\left(x_0 - Tx_0, x_0 - Tx_0 \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k^{2n-1}(\alpha,\beta)}\right)\right)\right)$$

$$= \Delta^{2m-2}\left(J\left(x_0 - Tx_0, x_0 - Tx_0 \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))^2}{k2^{n-1}(\alpha,\beta)}\right)\right),$$

$$J'\left(T^n x_0 - T^{m+n}x_0, T^n x_0 - T^{m+n}x_0 \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right)$$

$$\leq J'\left(x_0 - T^m x_0, T^n x_0 - T^{m+n}x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^{n+1}(\alpha,\beta)}\right)$$

$$\leq J'\left(x_0 - T^m x_0, x_0 - T^m x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^{2n-1}(\alpha,\beta)}\right)$$

$$\leq \nabla^{m-1}\left(J\left(x_0 - Tx_0, x_0 - T^m x_0 \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k^{2n+1}(\alpha,\beta)}\right)\right)$$

$$\leq \nabla^{m-1}\left(\nabla^{m-1}\left(J\left(x_0 - Tx_0, x_0 - Tx_0 \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))^2}{K^{2n+1}(\alpha,\beta)}\right)\right)\right)$$

$$= \nabla^{2m-1}\left(J\left(x_0 - Tx_0, x_0 - Tx_0 \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))^2}{k^{2n+1}(\alpha,\beta)}\right)\right)$$

Note that Δ,∇ are t-norm and t-conorm of h-type, the family of functions $\{\Delta^m(p)\}_{m=1}^{\infty}$ and $\{\Delta^m(p)\}_{m=1}^{\infty}$ is equi-continuous at p =1 and p = 0 respt. and the functions J, J' are non-decreasing, non-increasing with:

$$\lim_{t\to\infty} J(x, y \mid x_1,....,x_{n-1}, t) =$$
$$1 \text{ and } \lim_{t\to\infty} J'(x, y \mid x_1,...,x_{n-1}, t) = 0,$$

Then we have:

$$\lim_{n\to\infty} J\left(T^n x_0 - T^{m+n}x_0, T^n x_0 - T^{m+n}x_0 \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right)$$

$$\geq \lim_{n\to\infty} \Delta^{2m-2}\left(J\left(x_0 - Tx_0, x_0 - Tx_0 \mid x_1,...,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k^{2n+1}(\alpha,\beta)}\right)\right)$$

$$\lim_{n\to\infty} J'\left(T^n x_0 - T^{m+n}x_0, T^n x_0 - T^{m+n}x_0 \mid x_1,..,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right)$$

$$\leq \lim_{n\to\infty} \nabla^{2m-2}\left(J'\left(x_0 - Tx_0, x_0 - Tx_0 \mid x_1,..,x_{n-1}, \frac{t(1-k(\alpha,\beta))}{k^{2n+1}(\alpha,\beta)}\right)\right)$$

By ((2),(3)) and ((11,12)) of Definition 11, we have $\{T^m x_0\}_{m=1}^{\infty}$ is a Cauchy sequence in X. By the completeness of X, let $x_m \to x_* \in X$ ($m\to\infty$).

Secondly, we prove that $*\,x$ is a fixed point of T. Because of:

Then we have:

$$\lim_{i\to\infty} J\left(x_* - Tx_i, x_* - Tx_* \mid x_1,...,x_{n-1} \frac{t}{k(\alpha,\beta)}\right) = 1,$$

$$\lim_{i\to\infty} J'\left(x_* - Tx_i, x_* - Tx_* \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right) = 0 \forall t > 0.$$

Because $x_i \to x_*$ (when $i\to$) Δ D(•,•) and ∇ (•,•) are equi-continuous at (1,1) and (0, 0) respectively and:

$$J(\theta, x_* - Tx_* \mid x_1,...,x_{n-1}, t) = 1 \text{ and}$$
$$j'(\theta, x_* - Tx_* - Tx_* \mid x_1,...,x_{n-1}, t) = 0 \text{ we have}$$
$$\lim_{i\to} J\left(x_* - Tx_i, x_* - Tx_* \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right) = 1$$
$$\lim_{i\to\infty} J'\left(x_* - Tx_i, x_* - Tx_* \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right) = 0, \forall t > 0$$

Hence:

$$J\left(x_* - Tx_*, x_* - Tx_* \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right)$$

$$\geq \Delta\left(\begin{array}{l} J\left(x_* - Tx_i, x_* - Tx_* \mid x_1,...,x_{n-1}, \dfrac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right) \\ , J(Tx_i - Tx_*, x_* - Tx_* \mid x_1,...,x_{n-1}, t) \end{array}\right)$$

$$\geq \Delta\left(\begin{array}{l} J\left(x_* - Tx_i, x_* - Tx_* \mid x_1,...,x_{n-1}, \dfrac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right) \\ , J\left(x_i - x_* x_* - Tx_* \mid x_1,...,x_{n-1}, \dfrac{t}{k(\alpha,\beta)}\right) \end{array}\right),$$

$$J'\left(x_* - Tx_*, x_* - Tx_* \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right)$$

$$\leq \nabla\left(\begin{array}{l} J'\left(x_* - Tx_i, x_* - Tx_* \mid x_1,...,x_{n-1}, \dfrac{t(1-k(a,\beta))}{k(\alpha,\beta)}\right) \\ , J'(Tx_i - Tx_*, x_* - Tx_* \mid x_1,...,x_{n-1}, t) \end{array}\right)$$

$$\leq \nabla\left(\begin{array}{l} J'\left(x_* - Tx_i, x_* - Tx_* \mid x_1,...,x_{n-1}, \dfrac{t(1-k(\alpha,\beta))}{k(\alpha,\beta)}\right) \\ , J'\left(x_i - x_*, x_* - Tx_* \mid x_1,...,x_{n-1}, \dfrac{t}{k(\alpha,\beta)}\right) \end{array}\right)$$

$$\text{So}\, J\left(x_* - Tx_*, x_* - Tx_* \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right) \to 1,$$

$$J'\left(x_* Tx_*, x_* - Tx_* \mid x_1,...,x_{n-1}, \frac{t}{k(\alpha,\beta)}\right) \to 0(i \to \infty \forall t > 0).$$

By ((2),(3)) and ((11,12)) of Definition 11, we have $* * x = Tx$

If there exists a point $y \in X *$ such that $* * y = Ty$, then:

$$J(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1}, t) =$$
$$J(Tx_* - Tx_* - Ty_* \mid x_1,...,x_{n-1}, t)$$
$$\geq J\left(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1}, \frac{t}{(\alpha,\beta)}\right),$$
$$J'(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1}, t) = J'$$
$$(Tx_* - Ty_*, Tx_* - Ty_* \mid x_1,...,x_{n-1}, t)$$
$$\leq J'\left(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1}, \frac{t}{k^2(\alpha,\beta)}\right)$$

In the same way, we obtain:

$$J(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1}, t) \geq J$$
$$\left(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1}, \frac{t}{k^2(\alpha,\beta)}\right) \geq ... \geq$$
$$J\left(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1}, \frac{t}{k^{2n}(\alpha,\beta)}\right),$$
$$J'(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1}) \leq J'$$
$$\left(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1} \frac{t}{k^2(\alpha,\beta)}\right),$$
$$\leq J'\left(x_* - y_*, x_* - y_* \mid x_1,...,x_{n-1} \frac{t}{k^{2n}(\alpha,\beta)}\right).$$

Definition 11, we have $x_* = y_*$. Therefore x_* is the unique fixed point in X. Finally, we prove that the sequence $\{T^n x_0\}\tau$- converges to x_* for any $x_0 \in X$. Because of:

$$J(x_* - T^n x_0, x_* - T^n x_0 \mid x_1,...,x_{n-1}, t)$$
$$= J(Tx_* - T^n x_0, Tx_* - T^n x_0 \mid x_1,...,x_{n-1}, t)$$
$$\geq J\left(x_* - T^{n-1}x_0, x_* - T^{n-1}x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^2(\alpha,\beta)}\right)$$
$$= J\left(Tx_* - T^{n-1}x_0 Tx_* - \mid T^{n-1}x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^2(\alpha,\beta)}\right)$$
$$\geq \left(x_* - T^{n-2}x_0, x_* - T^{n-2}x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^4(\alpha,\beta)}\right)$$
$$\geq .... \geq J\left(x_* - x_0, x_* - x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^{2n}(\alpha,\beta)}\right)$$
$$J'(x_* - T^n x_0, x_* - T^n x_0 \mid x_1,...,x_{n-1}, t)$$
$$= J'(Tx_* - T^n x_0, Tx_* - T^n x_0 \mid x_1,...,x_{n-1}, t)$$
$$\leq J'\left(x_* - T^{n-1}x_0, x_* - T^{n-1}x_0 \mid x_1,..,x_{n-1}, \frac{t}{k^2(\alpha,\beta)}\right)$$
$$= J'\left(Tx_* - T^{n-1}x_0, Tx_* - T^{n-1}x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^2(\alpha,\beta)}\right)$$
$$\leq J'\left(x_* - T^{n-2}x_0, x_* - T^{n-2}x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^4(\alpha,\beta)}\right)$$
$$\leq ... \leq J'\left(x_* - x_0, x_* - x_0 \mid x_1,...,x_{n-1}, \frac{t}{k^{2n}(\alpha,\beta)}\right)$$

We have:

$$\lim_{n\to\infty} J(x_* - T^n x_0, x_* - T^n x_0 \mid x_1,...,x_{n-1}, t)$$
$$\geq \lim_{n\to\infty} J\left(x_* - x_0, x_* - x_0 \mid x_1,....,x_{n-1}, \frac{t}{k^{2n}(\alpha,\beta)}\right) = 1$$
$$\lim_{n\to\infty} J'(x_* - T^n x_0, x_* - T^n x_0 \mid x_1,...,x_{n-1}, t)$$
$$\leq \lim_{n\to\infty} J'\left(x_* - x_0, x_* - x_0 \mid x_1,....,x_{n-1}, \frac{t}{k^{2n}(\alpha,\beta)}\right) = 0$$

Applications: In what follows we shall utilize theorem 11 to study the existence and uniqueness of solution for linear Volterra integral equation on complete IF-n-IP space. We assume that [a,b] is a fixed real interval. We define linear operation in L^2 [a, b]:

$$(x+y)(t)=x(t)+y(t),(ax)(t)=ax(t)$$

Then L^2 [a, b] is a linear space. We lead n-inner product into L^2 [a, b]:

$$(x,y\mid x_1,\ldots,x_{n-1})=\int_a^b x(t).y(t).x_1(t).x_2(t)\ldots x_{n-1}(t)dt$$

$$\text{Hence}\,(x,y\mid x_1,\ldots,x_{n-1})\,\text{is finite}$$

Number, (• , • | • , . . . , •) satisfies all conditions of inner product and L^2 [a, b] is an n-inner product space by (• , • | • , . . . , •). Because L^2 [a, b] is infinite dimensional and complete, then L^2 [a, b] is a Hilbert space. Define a space (L^2 [a, b], J, J', Δ,∇), where:

$$J,J':\overbrace{L^2[a,b]\times L^2[a,b]\times\ldots\times L^2[a,b]}^{n+1\,\text{times}}\to[0,1]$$

$$J(x,y\mid x_1,\ldots,x_{n-1},t)=\frac{t}{t+(x,y\mid x_1,\ldots,x_{n-1})}\,\text{and}$$

$$J'(x,y\mid x_1,\ldots,x_{n-1},t)=\frac{(x,y\mid x_1,\ldots,x_{n-1})}{t+(x,y\mid x_1,\ldots,x_{n-1})}$$

Then (L^2 [a, b], J, J', Δ,∇) is a IFIP-space. In fact, let $\{x_n\}$ be a Cauchy sequence in (L^2 [a, b], J, J', Δ,∇). Then for any $\varepsilon > 0,\lambda \in (0,1],\exists N$, when m, n ≥ N, we have:

$$J(x_m-x_n,x_m-x_n\mid x_1,\ldots,x_{n-1},\varepsilon)>$$
$$1-\lambda,J'(x_m-x_n,x_m\mid x_1,\ldots,x_{n-1},\varepsilon)\le\lambda$$

Because of:

$$J(x_m-x_n,x_m-x_n\mid x_1,\ldots,x_{n-1},\varepsilon)=\frac{\varepsilon}{\varepsilon+(x_m-x_n,x_m-x_n\mid x_1,\ldots,x_{n-1})}$$
$$=\frac{\varepsilon}{\varepsilon+\int_a^b(x_m-x_n)(t).(x_m-x_n)(t).(x_1)(t).(x_2)(t)\ldots.(x_{n-1})(t)dt}$$
$$>1-\lambda$$

And:

$$J'(x_m-x_n,x_m-x_n\mid x_1,\ldots,x_{n-1},\varepsilon)=\frac{(x_m-x_n,x_m-x_n\mid x_1,\ldots,x_{n-1})}{\varepsilon+(x_m-x_n,x_m-x_n\mid x_1,\ldots,x_{n-1})}$$
$$=\frac{\int_a^b(x_m-x_n)(t)(x_m-x_n)(t).(x_1)(t)(x_2)(t)\ldots.(x_{n-1})(t)dt}{\varepsilon+\int_a^b(x_m-x_n)(t)(x_m-x_n)(t).(x_1)(t).(x_2)(t)\ldots.(x_{n-1})(t)dt}\le\lambda$$

We have:

$$\int_a^b[(x_m-x_n)(t)]^2.(x_1)(t).(x_2)(t)\ldots.(x_{n-1})(t)dt\to0.$$

Then $x_m - x_n \to 0$ a.e. as $x_1, x_2,\ldots, x_{n-1}$ are not all zero. So x_n is a Cauchy sequence in L^2 [a, b]. By the completeness of L^2 [a, b], we have $x_n \to x_* \in L^2$ [a, b]. Hence $x_* \in$ (L^2 [a, b], J, J', Δ,∇). So(L^2 [a, b], J , J' , Δ,∇) is a complete IF-n-IP space.

Theorem 15: Let (L^2 [a, b], J, J', Δ,∇) be a complete IF-n-IP space. Then the following conditions are satisfied:

$$\int_a^t(x(s)-y(s))ds\le x(t)-y(t),\forall x(.),y(.)\in L^2[a,b]$$

Let T be a linear mapping and defined as follows:

$$(Tx)(t)=f(t)+\lambda\int_a^t k(t,s)x(s)ds$$

where $f \in L^2$ [a, b] is a given function, k(t, s) is a continuous function defined on $a \le t \le b$, $a \le s \le t$, χ is a constant, we denote:

$$\max_{a\le t\le b,a\le s\le t} k(t,s)=M$$

Then when $\chi M \in (0, 1)$, T has a unique fixed point in L^2 [a, b]. Furthermore, for any $x_0 \in L^2$ [a, b], the iterative sequence $\{T^n x_0\}\tau-$ converges to the fixed point.

Proof:

$$(Tx-Ty,v\mid x_1,\ldots,x_{n-1})=$$
$$\int_a^b(Tx-Ty)(t).(v)(t).(x_1)(t).(x_2)(t)\ldots.(x_{n-1})(t)dt$$
$$=\int_a^b\lambda\left(\int_a^t\left(k(t,s)x(s)ds-\int_a^t k(t,s)y(s)ds\right)\right)$$
$$v(t).(x_1)(t).(x_2)(t)\ldots.(x_{n-1})(t)dt$$
$$=\lambda\int_a^b\left(\int_a^t k(t,s)(x(s)-y(s))ds\right)$$
$$v(t).(x_1)(t).(x_2)(t)\ldots.(x_{n-1})(t)dt$$

By the continuity of k (t, s) and mean value theorem, $\exists t_1, s_1$, we have:

$$(Tx - Ty, v \mid x_1, ..., x_{n-1}) = \lambda k(t_1, s_1) \int_a^b \left(\int_a^t (x(s) - y(s))ds \right)$$
$$v(t).(x_1)(t).(x_2)(t)....(x_{n-1})(t)dt$$

By the condition (i), we have:

$$(Tx - Ty, v \mid x_1, ..., x_{n-1}) \le \lambda k \int_a^b \left(\int_a^t (x(s) - y(s))ds \right)$$
$$v(t).(x_1)(t).(x_2)(t)...(x_{n-1})\ (t)dt$$
$$= \lambda M(x - y, v \mid x_1, ..., x_{n-1})$$

Therefore:

$$(Tx - Ty, v \mid x_1, ..., x_{n-1}) \le \lambda M(x - y, v \mid x_1, ..., x_{n-1})$$
$$\frac{t}{t + (Tx - Ty, v \mid x_1, ..., x_{n-1})} \ge \frac{t}{t + \lambda M(x - y, v \mid x_1, ..., x_{n-1})}$$

Because of $\chi M \in (0,1)$, we get:

$$\frac{t}{t + (Tx - Ty, v \mid x_1, ..., x_{n-1})} \ge \frac{\frac{t}{\lambda M}}{\frac{t}{\lambda M} + (x - \mid y, v \mid x_1, ..., x_{n-1})}$$
$$J(Tx - Ty, v \mid x_1, ..., X_{n-1}) \ge J\left(x - y, v \mid x_1, ..., x_{n-1}, \frac{t}{\lambda M} \right) \text{and}$$
$$(Tx - Ty, v \mid x_1, ..., x_{n-1}) \le \lambda M\left(x - y, v \mid x_1, ..., x_{n-1} \right)$$
$$\frac{t}{(Tx - Ty, v \mid x_1, ..., x_{n-1})} \ge \frac{t}{\lambda M(x - y \mid x_1, ..., x_{n-1})}$$
$$1 + \frac{t}{(Tx - Ty, v \mid x_1, ..., x_{n-1})} \ge 1 + \frac{t}{\lambda M(x - y \mid x_1, ..., x_{n-1})}$$
$$\frac{(Tx - Ty, v \mid x_1, ..., x_{n-1}) + t}{(Tx - Ty, v \mid x_1, .., x_{n-1})} \ge \frac{\lambda M\left(x - y, v \mid x_1, ..., x_{n-1} \right) + t}{\lambda M(x - y, v \mid x_1, ..., x_{n-1})}$$
$$\frac{(Tx - Ty, v \mid x_1, ..., x_{n-1})}{t + (Tx - Ty, v \mid x_1, ..., x_{n-1})} \le \frac{\lambda M\left(x - y, v \mid x_1, ..., x_{n-1} \right)}{t + \lambda M(x - y, v \mid x_1, ..., x_{n-1})}$$
$$\frac{(Tx - Ty, v \mid x_1, ..., x_{n-1})}{t + (Tx - Ty, v \mid x_1, ..., x_{n-1})} \le \frac{\lambda M(x - y, v \mid x_1, ..., x_{n-1})}{t + \lambda M(x - y, v \mid x_1, ..., x_{n-1})}$$
$$= \frac{(x - y, v \mid x_1, ..., x_{n-1})}{\frac{t}{\lambda M} + (x - y, v \mid x_1, ..., x_{n-1})}$$
$$J'(T_x - Ty, v \mid x_1, ..., x_{n-1}, t) J'\left(x - y, v \mid x_1, ..., x_{n-1}, \frac{t}{\lambda M} \right)$$

By Theorem 11, T has a unique fixed point in L^2 [a, b]. Furthermore, for any $x_0 \in L^2$ [a, b], the iterative sequence $\left\{T^m x_0\right\}\tau-$ converges to the fixed point. This completes the proof.

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