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Number of Spanning Trees of Circulant Graphs C_{6n} and their Applications

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Abstract: Problem statement: The number of spanning trees of a graph G is usually denoted by $\tau(G)$. A circulant graph with n vertices and k jumps is C_n $(a_1,\ldots,a_k)$. **Approach:** In this study the number $\tau(G)$ of spanning trees of the circulant graphs C_{6n} with some non-fixed jumps such as C_{6n} (1, n), C_{6n} (1, n, 2n), C_{6n} (1, n, 3n), C_{6n} (1, 2n 3n), C_{6n} (1, n, 2n, 3n), are evaluated using Chebyshev polynomials. A large number of theorems of number of the spanning trees of circulate graphs C_{12n} are obtained. **Results:** The number $\tau(G)$ of spanning trees of the circulant graphs C_{6n}(1, n), C_{6n}(1, n, 2n), C_{6n}(1, n, 3n), C_{6n}(1, 2n, 3n), C_{6n}(1, n, 2n, 3n), C_{12n}(1, 2n, 3n), C_{12n}(1, 3n, 6n), C_{12n}(1, 3n, 4n), C_{12n}(1, 2n, 3n, 4n), C_{12n}(1, 2n, 3n, 6n), C_{12n}(1, 3n, 4n, 6n) and C_{12n}(1, 2n, 3n, 4n, 6n) are evaluated. **Conclusion:** The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number and analyzing its behavior is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the following important theorems and their proofs.

Key words: Spanning trees, circulant graphs, chebyshev polynomials, kirchhoff matrix

INTRODUCTION

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number and analyzing its behavior is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees.

In this study we consider finite undirected graph with no loops or multiple edges. Let G be such a graph of n vertices. A spanning tree for a graph G is a sub graph of G that is a tree and contains all vertices of G .The number of spanning trees of G, denoted by $\tau(G)$, is the total number of distinct spanning sub graphs of G that are trees. A classic result of Kirchhoff (1874) an be used to determine the number of spanning trees for G = (V, E). Let $V = (v_1,v_2,\ldots,v_n)$. The Kirchhoff matrix H is defined as n×n characteristic matrix H = D-A, where, D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G, $H = [a_{ij}]$ is defined as follows: (i) $a_{ij} = -1$, when v_i and v_j are adjacent and $i\neq j$, (ii) a_{ij} equal the degree of vertex v_i if $i\neq j$ and (iii) $a_{ij} = 0$ otherwise. All of co-factors of H are equal to $\tau(G)$. There are more than method for calculating $\tau(G)$.

Let $\mu_1\geq\mu_1\geq\ldots\geq\mu_p$ denote the eignvalues of H matrix of a p point graph. It can be easily shown that $\mu p = 0$. Kelmans and Chelnokov (1974) proved that Eq. 1:

$$\tau(G) = \frac{1}{p}\prod_{k=1}^{p-1}\mu_k \tag{1}$$

The formula for the number of spanning trees in a d-regular graph G can be expressed as $\tau(G) = \frac{1}{p}\prod_{k=1}^{p-1}(d-\mu_k)$ where $\mu_0 = d, \mu_1, \mu_2,\ldots,\mu_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. The circulant graphs are an important class of graphs, which are used in the design of local area networks.

Let $1 \le a_1 \le a_2 \le ..... \le a_k \le \frac{n}{2}$, where n and a_i(i = 1,2…,k) are positive integers. An undirected circulant graph $C_n(a_1,a_2,…,a_k)$ is a regular graph whose set of vertices is V = {0,1,2,…,n-1} and whose set of edges is E = {(i, i+a_j(mod n))/i = 0,1,2,…,n-1, j = 1,2,…,k}.

If $a_k \le \frac{n}{2}$, then $C_n(a_1,a_2,…,a_k)$ is a 2k-regular graph; if $a_k = \frac{n}{2}$, then it is a (2k-1)-regular one, Nikolopoulos and Papadopoulos (2004). The simplest circulant graph is the n vertex cycle C_n (1). The well known formula $\tau(C_n (1, 2)) = nF_n^2$, where Fn is the $n^{\underline{th}}$ Fibonacci number defined by the recursive relation $F_n = F_{n-1}+F_{n-2}$, n = 2, 3….with initial condition F_0 = 0, F = 1, Kleiman and Golden (1975). The formulas for $\tau(C_n (1, 3))$, $\tau(C_n(1,4))$ and more general results have recently been obtained by Yong and Atajan (1997). The formulas of $\tau(C_{2n} (1, n))$, $\tau(C_{3n}(1,n))$, $\tau(C_{4n}(1,n))$ can be found in Zhanga *et al*. (2005).

Chebyshev polynomial: Now we introduce some some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations.

We begin from their definitions, Zhanga *et al*. (2000).

Let $A_n(x)$ be n×n matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & & \\ -1 & 2x & -1 & 0 & \\ 0 & \ddots & \ddots & & \ddots \\ & \ddots & \ddots & \ddots & -1 \\ & & 0 & -1 & 2x \end{pmatrix}$$

where all other elements are zeros.

Further we recall that the Chebyshev polynomials of the first kind are defined by Eq. 2:

$$T_n(x) = \cos(n \arccos x) \tag{2}$$

The Chebyshev polynomials of the second kind are defined by Eq. 3:

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)} \tag{3}$$

It is easily verified that Eq. 4:

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \tag{4}$$

It can then be shown from this recursion that by expanding det A_n (x) one gets Eq. 5:

$$U_n(x) = \det(A_n(x)), n \ge 1 \tag{5}$$

Furthermore by using standard methods for solving the recursion (4), one obtains the explicit formula Eq. 6:

$$\begin{aligned} U_n(x) = \frac{1}{2\sqrt{x^2-1}}[(x+\sqrt{x^2-1})^{n+1} \\ -(x-\sqrt{x^2-1})^{n+1}], n \ge 1 \end{aligned} \tag{6}$$

where, the identity is true for all complex x (except at x = ±1where the function can be taken as the limit).

The definition of $U_n(x)$ easily yields its zeros and it can therefore be verified that Eq. 7:

$$U_{n-1}(x) = 2^{n-1}\prod_{j=1}^{n-1}(x - \cos\frac{j\pi}{n}) \tag{7}$$

One further notes that Eq. 8:

$$U_{n-1}(-x) = (-1)^{n-1}U_{n-1}(x) \tag{8}$$

These two results yield another formula for $U_n(x)$:

$$U_{n-1}^2(x) = 4^{n-1}\prod_{j=1}^{n-1}(x^2 - \cos^2\frac{j\pi}{n}) \tag{9}$$

Finally, simple manipulation of the above formula yields the following, which also will be extremely useful to us latter Eq. 9-12:

$$U_{n-1}^2(\sqrt{\frac{x+2}{4}}) = \prod_{j=1}^{n-1}(x - 2\cos\frac{2j\pi}{n}) \tag{10}$$

Furthermore one can show that:

$$\begin{aligned} U_{n-1}^2(x) = \frac{1}{2(1-x^2)}[1-T_{2n}] = \\ \frac{1}{2(1-x^2)}[1-T_n(2x^2-1)] \end{aligned} \tag{11}$$

And:

$$T_n(x) = \frac{1}{2}[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n] \tag{12}$$

Lemma 1: The Kirchhoff matrix of the circulant graph C_n $(a_1,a_2,\dots,a_k)$ has n eigenvalues. They are 0 and $\forall j, 1 \le j \le n-1$ the values:

$$2k-\varepsilon^{-s_1 j}-\dots\dots-\varepsilon^{-s_k j}-\varepsilon^{s_1 j}-\dots\dots-\varepsilon^{s_k j}$$

where, $\varepsilon = e^{\frac{2\pi i}{n}}$ (Biggs, 1993).

Plugging this into(1) yields:

Corollary 2:

$$\tau(C_n(s_1,s_2,\dots\dots,s_k)) = \frac{1}{n}\prod_{j=1}^{n-1}(2k-\varepsilon^{-s_1 j}-\dots\dots-\varepsilon^{-s_k j}-\varepsilon^{s_1 j}-\dots\dots-\varepsilon^{s_k j})$$
$$= \frac{1}{n}\prod_{j=1}^{n-1}\left(\sum_{i=1}^{k}2-2\cos\frac{2js_i\pi}{n}\right)$$

Theorem 3:

$$\tau(C_{6n}(1,n)) = \frac{n}{6}\left[\left(\sqrt{\frac{5}{4}}+\sqrt{\frac{1}{4}}\right)^{3n}+\left(\sqrt{\frac{5}{4}}-\sqrt{\frac{1}{4}}\right)^{3n}\right]^2\times\left[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n\right]^2$$
$$\times\left[\left(\sqrt{\frac{5}{4}}+\sqrt{\frac{1}{4}}\right)^{n}+\left(\sqrt{\frac{5}{4}}-\sqrt{\frac{1}{4}}\right)^{n}\right]^{-2}$$
$$\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2$$

Proof: Let $\varepsilon = e^{2\pi i/6n}$. By Lemma 2.1 we have:

$$\tau(C_{6n}(1,n)) = \frac{1}{6n}\prod_{j=1}^{6n-1}[4-\varepsilon^{-j}-\varepsilon^{-nj}-\varepsilon^{j}-\varepsilon^{nj}]$$
$$= \frac{1}{6n}\prod_{j=1}^{6n-1}\left[4-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi jn}{6n}\right]$$
$$= \frac{1}{6n}\prod_{j=1}^{6n-1}\left[4-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}\right]$$
$$= \frac{1}{6n}\prod_{j=1,2\nmid j,3\nmid j}^{6n-1}\left[3-2\cos\frac{2\pi j}{6n}\right]\times\prod_{j=1,2|j,3|j}^{6n-1}\left[2-2\cos\frac{2\pi j}{6n}\right]$$
$$\times\prod_{j=1,2\nmid j,3|j}^{6n-1}\left[6-2\cos\frac{2\pi j}{6n}\right]\times\prod_{j=1,2|j,3\nmid j}^{6n-1}\left[5-2\cos\frac{2\pi j}{6n}\right]$$
$$= \frac{1}{6n}\prod_{j=1}^{6n-1}\left[3-2\cos\frac{2\pi j}{6n}\right]\times\prod_{j=1}^{6n-1}\frac{(3-2\cos\frac{2\pi j}{6n})(2-2\cos\frac{2\pi j}{6n})}{(6-2\cos\frac{2\pi j}{6n})(5-2\cos\frac{2\pi j}{6n})}$$
$$\times\prod_{j=1}^{6n-1}\frac{(5-2\cos\frac{2\pi j}{6n})}{(3-2\cos\frac{2\pi j}{6n})}\times\prod_{j=1}^{6n-1}\frac{(6-2\cos\frac{2\pi j}{6n})}{(3-2\cos\frac{2\pi j}{6n})}$$

Put $j = 6j'$ in the second term, $j = 2j'$ in the third term and $j = 3j'$ in the fourth term for some integer j', we get:

$$\tau(C_{6n}(1,n)) = \frac{1}{6n}\prod_{j=1}^{6n-1}\left[3-2\cos\frac{2\pi j}{6n}\right]\times$$
$$\prod_{j=1}^{n-1}\frac{(3-2\cos\frac{2\pi j}{n})(2-2\cos\frac{2\pi j}{n})}{(6-2\cos\frac{2\pi j}{n})(5-2\cos\frac{2\pi j}{n})}$$
$$\times\prod_{j=1}^{3n-1}\frac{(5-2\cos\frac{2\pi j}{3n})}{(3-2\cos\frac{2\pi j}{3n})}\times\prod_{j=1}^{2n-1}\frac{(6-2\cos\frac{2\pi j}{2n})}{(3-2\cos\frac{2\pi j}{2n})}$$
$$= \frac{1}{6n}\frac{U^2_{6n-1}(\sqrt{\frac{5}{4}})\cdot U^2_{n-1}(\sqrt{\frac{5}{4}})\cdot U^2_{n-1}(1)\cdot U^2_{3n-1}(\sqrt{\frac{7}{4}})\cdot U^2_{2n-1}(\sqrt{2})}{U^2_{n-1}(\sqrt{2})\cdot U^2_{n-1}(\sqrt{\frac{7}{4}})\cdot U^2_{3n-1}(\sqrt{\frac{5}{4}})\cdot U^2_{2n-1}(\sqrt{\frac{5}{4}})}$$
$$= \frac{n}{6}\left[\left(\sqrt{\frac{5}{4}}+\sqrt{\frac{1}{4}}\right)^{3n}+\left(\sqrt{\frac{5}{4}}-\sqrt{\frac{1}{4}}\right)^{3n}\right]^2\times\left[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n\right]^2$$
$$\times\left[\left(\sqrt{\frac{5}{4}}+\sqrt{\frac{1}{4}}\right)^{n}+\left(\sqrt{\frac{5}{4}}-\sqrt{\frac{1}{4}}\right)^{n}\right]^{-2}\times\left[\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^{2n}+\left(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}}\right)^{2n}+1\right]^2$$

where, are used to derive the last two steps.

Theorem 4:

$$\tau(C_{6n}(1,n,2n)) = \frac{n}{6}\left[(\sqrt{2}+1)^{3n}+(\sqrt{2}-1)^{3n}\right]^2$$
$$\times\left[\left(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}}\right)^{2n}+\left(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}}\right)^{2n}+1\right]^2$$

Proof: Let $\varepsilon = e^{2\pi i/6n}$. By Lemma 2.1 we have:

$$\tau(C_{6n}(1,n,2n)) = \frac{1}{6n}\prod_{j=1}^{6n-1}[6-\varepsilon^{-j}-\varepsilon^{-nj}-\varepsilon^{-2nj}-\varepsilon^{j}-\varepsilon^{nj}-\varepsilon^{2nj}]$$
$$= \frac{1}{6n}\prod_{j=1}^{6n-1}\left[6-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi jn}{6n}-2\cos\frac{4\pi jn}{6n}\right]$$
$$= \frac{1}{6n}\prod_{j=1}^{6n-1}\left[6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}-2\cos\frac{2\pi j}{3}\right]$$
$$= \frac{1}{6n}\prod_{j=1,3\nmid j}^{6n-1}\left[7-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}\right]\times\prod_{j=1,3|j}^{6n-1}\left[4-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}\right]$$
$$= \frac{1}{6n}\prod_{j=1}^{6n-1}\left[7-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}\right]\times\prod_{j=1}^{6n-1}\frac{4-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}}{7-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}}$$

Put $j = 3j'$ in the second term for some integer j', we get:

$$\tau(C_{6n}(1,n,2n)) = \frac{1}{6n}\prod_{j=1}^{6n-1}[7-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}]\times$$

$$\prod_{j=1}^{2n-1}\frac{4-2\cos\frac{2\pi j}{2n}-2\cos\pi j}{7-2\cos\frac{2\pi j}{2n}-2\cos\pi j} = \frac{1}{6n}\prod_{j=1,2\nmid j,3\nmid j}^{6n-1}[6-2\cos\frac{2\pi j}{6n}]$$

$$\times\prod_{j=1,2|j,3|j}^{6n-1}[5-2\cos\frac{2\pi j}{6n}]\times\prod_{j=1,2\nmid j,3|j}^{6n-1}[9-2\cos\frac{2\pi j}{6n}]\times$$

$$\prod_{j=1,2|j,3\nmid j}^{6n-1}[8-2\cos\frac{2\pi j}{6n}]\times$$

$$\frac{\prod_{j=1,2\nmid j}^{2n-1}[6-2\cos\frac{2\pi j}{2n}]\times\prod_{j=1,2|j}^{2n-1}[2-2\cos\frac{2\pi j}{2n}]}{\prod_{j=1,2\nmid j}^{2n-1}[9-2\cos\frac{2\pi j}{2n}]\times\prod_{j=1,2|j}^{2n-1}[5-2\cos\frac{2\pi j}{2n}]} = \frac{1}{6n}$$

$$\prod_{j=1}^{6n-1}[6-2\cos\frac{2\pi j}{6n}]\times$$

$$\prod_{j=1}^{6n-1}\frac{(6-2\cos\frac{2\pi j}{6n})(5-2\cos\frac{2\pi j}{6n})}{(9-2\cos\frac{2\pi j}{6n})(8-2\cos\frac{2\pi j}{6n})}\times\prod_{j=1}^{6n-1}\frac{(8-2\cos\frac{2\pi j}{6n})}{(6-2\cos\frac{2\pi j}{6n})}$$

$$\times\prod_{j=1}^{6n-1}\frac{(9-2\cos\frac{2\pi j}{6n})}{(6-2\cos\frac{2\pi j}{6n})}]\times\frac{\prod_{j=1}^{2n-1}[6-2\cos\frac{2\pi j}{2n}]}{\prod_{j=1}^{2n-1}[9-2\cos\frac{2\pi j}{2n}]}\times\frac{\prod_{j=1}^{2n-1}\frac{2-2\cos\frac{2\pi j}{2n}}{6-2\cos\frac{2\pi j}{2n}}}{\prod_{j=1}^{2n-1}\frac{5-2\cos\frac{2\pi j}{2n}}{9-2\cos\frac{2\pi j}{2n}}}$$

Put j = 6 j’ in the second term, j = 2 j’ in the third term, j = 3 j’ in the fourth term and j = 2 j’ in the sixth term for some integer j’, we get:

$$\tau(C_{6n}(1,n,2n)) = \frac{1}{6n}\prod_{j=1}^{6n-1}[6-2\cos\frac{2\pi j}{6n}]\times$$

$$\prod_{j=1}^{n-1}\frac{(6-2\cos\frac{2\pi j}{n})(5-2\cos\frac{2\pi j}{n})}{(9-2\cos\frac{2\pi j}{n})(8-2\cos\frac{2\pi j}{n})}$$

$$\times\prod_{j=1}^{3n-1}\frac{(8-2\cos\frac{2\pi j}{3n})}{(6-2\cos\frac{2\pi j}{3n})}\times\prod_{j=1}^{2n-1}\frac{(9-2\cos\frac{2\pi j}{2n})}{(6-2\cos\frac{2\pi j}{2n})}]\times\frac{\prod_{j=1}^{2n-1}[6-2\cos\frac{2\pi j}{2n}]}{\prod_{j=1}^{2n-1}[9-2\cos\frac{2\pi j}{2n}]}$$

$$\times\frac{\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{6-2\cos\frac{2\pi j}{n}}}{\prod_{j=1}^{n-1}\frac{5-2\cos\frac{2\pi j}{n}}{9-2\cos\frac{2\pi j}{n}}} = \frac{n}{6}\frac{U_{6n-1}^2(\sqrt{2})\cdot U_{3n-1}^2(\sqrt{\frac{5}{2}})}{U_{3n-1}^2(\sqrt{2})\cdot U_{n-1}^2(\sqrt{\frac{5}{2}})} = \frac{n}{6}[(\sqrt{2}+1)^{3n}$$

$$+(\sqrt{2}-1)^{3n}]^2\times[(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}})^{2n}+(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})^{2n}+1]^2$$

Theorem 5:

$$\tau(C_{6n}(1,n,3n)) = \frac{n}{6}[(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}})^{2n}+(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}})^{2n}-1]^2$$

$$\times[(\sqrt{3}+\sqrt{2})^n+(\sqrt{3}-\sqrt{2})^n]^2$$

$$\times[(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}})^{2n}+(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}})^{2n}+1]^2$$

Proof: Let $\varepsilon = e^{2\pi i/6n}$. By Lemma 2.1 we have:

$$\tau(C_{6n}(1,n,3n)) = \frac{1}{6n}\prod_{j=1}^{6n-1}[6-\varepsilon^{-j}-\varepsilon^{-nj}-\varepsilon^{-3nj}-\varepsilon^{j}-\varepsilon^{nj}-\varepsilon^{3nj}]$$

$$= \frac{1}{6n}\prod_{j=1}^{6n-1}[6-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi jn}{6n}-2\cos\frac{6\pi jn}{6n}]$$

$$= \frac{1}{6n}\prod_{j=1}^{6n-1}[6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}-2\cos\pi j]$$

$$= \frac{1}{6n}\prod_{j=1,2\nmid j}^{6n-1}[8-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}]$$

$$\times\prod_{j=1,2|j}^{6n-1}[4-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}]$$

$$= \frac{1}{6n}\prod_{j=1}^{6n-1}[8-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}]$$

$$\times\prod_{j=1}^{6n-1}\frac{4-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}}{8-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}}$$

Put j = 2 j' in the second term for some integer j', we get:

$$\tau(C_{6n}(1,n,3n)) = \frac{1}{6n}\prod_{j=1}^{6n-1}[8-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}]\times$$

$$\prod_{j=1}^{3n-1}\frac{4-2\cos\frac{2\pi j}{3n}-2\cos\frac{2\pi j}{3}}{8-2\cos\frac{2\pi j}{3n}-2\cos\frac{2\pi j}{3}} = \frac{1}{6n}\prod_{j=1,2\nmid j,3\nmid j}^{6n-1}[7-2\cos\frac{2\pi j}{6n}]$$

$$\times\prod_{j=1,2|j,3|j}^{6n-1}[6-2\cos\frac{2\pi j}{6n}]\times\prod_{j=1,2\nmid j,3|j}^{6n-1}[10-2\cos\frac{2\pi j}{6n}]\times$$

$$\prod_{j=1,2|j,3\nmid j}^{6n-1}[9-2\cos\frac{2\pi j}{6n}]$$

$$\times\frac{\prod_{j=1,3\nmid j}^{3n-1}[5-2\cos\frac{2\pi j}{3n}]\times\prod_{j=1,3|j}^{3n-1}[2-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1,3\nmid j}^{3n-1}[9-2\cos\frac{2\pi j}{3n}]\times\prod_{j=1,3|j}^{3n-1}[6-2\cos\frac{2\pi j}{3n}]}$$

$$= \frac{1}{6n}\prod_{j=1}^{6n-1}[7-2\cos\frac{2\pi j}{6n}]$$

$$\times\prod_{j=1}^{6n-1}\frac{(7-2\cos\frac{2\pi j}{6n})(6-2\cos\frac{2\pi j}{6n})}{(10-2\cos\frac{2\pi j}{6n})(9-2\cos\frac{2\pi j}{6n})}$$

$$\times\prod_{j=1}^{6n-1}\frac{(9-2\cos\frac{2\pi j}{6n})}{(7-2\cos\frac{2\pi j}{6n})}\times\prod_{j=1}^{6n-1}\frac{(10-2\cos\frac{2\pi j}{6n})}{(7-2\cos\frac{2\pi j}{6n})}]$$

$$\times\frac{\prod_{j=1}^{3n-1}[5-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1}^{3n-1}[9-2\cos\frac{2\pi j}{3n}]}\frac{\prod_{j=1}^{3n-1}\frac{2-2\cos\frac{2\pi j}{3n}}{5-2\cos\frac{2\pi j}{3n}}}{\prod_{j=1}^{3n-1}\frac{6-2\cos\frac{2\pi j}{3n}}{9-2\cos\frac{2\pi j}{3n}}}$$

Put j = 6 j' in the second term, j = 2 j' in the third term, j = 3 j' in the fourth term and j = 3 j' in the sixth term for some integer j', we get:

$$=\frac{1}{6n}\prod_{j=1}^{6n-1}[7-2\cos\frac{2\pi j}{6n}]\times$$

$$\prod_{j=1}^{n-1}\frac{(7-2\cos\frac{2\pi j}{n})(6-2\cos\frac{2\pi j}{n})}{(10-2\cos\frac{2\pi j}{n})(9-2\cos\frac{2\pi j}{n})}\times\prod_{j=1}^{3n-1}\frac{(9-2\cos\frac{2\pi j}{3n})}{(7-2\cos\frac{2\pi j}{3n})}$$

$$\times\prod_{j=1}^{2n-1}\frac{(10-2\cos\frac{2\pi j}{2n})}{(7-2\cos\frac{2\pi j}{2n})}]\times\frac{\prod_{j=1}^{3n-1}[5-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1}^{3n-1}[9-2\cos\frac{2\pi j}{3n}]}$$

$$\times\frac{\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{5-2\cos\frac{2\pi j}{n}}}{\prod_{j=1}^{n-1}\frac{6-2\cos\frac{2\pi j}{n}}{9-2\cos\frac{2\pi j}{n}}}$$

$$=\frac{n}{6}\frac{U_{6n-1}^2(\sqrt{\frac{9}{4}})\cdot U_{n-1}^2(\sqrt{\frac{9}{4}})\cdot U_{2n-1}^2(\sqrt{3})\cdot U_{3n-1}^2(\sqrt{\frac{7}{4}})}{U_{3n-1}^2(\sqrt{\frac{9}{4}})\cdot U_{2n-1}^2(\sqrt{\frac{9}{4}})\cdot U_{n-1}^2(\sqrt{3})\cdot U_{n-1}^2(\sqrt{\frac{7}{4}})}$$

$$=\frac{n}{6}[(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}})^{2n}+(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}})^{2n}-1]^2\times[(\sqrt{3}+\sqrt{2})^n$$

$$+(\sqrt{3}-\sqrt{2})^n]^2\times[(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}})^{2n}+(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}})^{2n}+1]^2$$

Theorem 6:

$$\tau(C_{6n}(1,2n,3n))=\frac{n}{6}[(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}})^{2n}+(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}})^{2n}-1]^2$$

$$\times[(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}})^{2n}+(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}})^{2n}+1]^2\times[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n]^2$$

Proof: Let $\varepsilon=e^{2\pi i/6n}$. By Lemma 2.1 we have:

$$\tau(C_{6n}(1,2n,3n))=\frac{1}{6n}\prod_{j=1}^{6n-1}[6-\varepsilon^{-j}-\varepsilon^{-2nj}-\varepsilon^{-3nj}-\varepsilon^{j}-\varepsilon^{2nj}-\varepsilon^{3nj}]$$

$$=\frac{1}{6n}\prod_{j=1}^{6n-1}[6-2\cos\frac{2\pi j}{6n}-2\cos\frac{4\pi jn}{6n}-2\cos\frac{6\pi jn}{6n}]$$

$$=\frac{1}{6n}\prod_{j=1}^{6n-1}[6-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-2\cos\pi j]$$

$$=\frac{1}{6n}\prod_{j=1,2\nmid j}^{6n-1}[8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}]\times\prod_{j=1,2\mid j}^{6n-1}[4-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}]$$

$$=\frac{1}{6n}\prod_{j=1}^{6n-1}[8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}]\times\prod_{j=1}^{6n-1}\frac{4-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}}{8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}}$$

Put j = 2 j' in the second term for some integer j', we get:

$$\tau(C_{6n}(1,2n,3n))=\frac{1}{6n}\prod_{j=1}^{6n-1}[8-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}]$$

$$\times\prod_{j=1}^{3n-1}\frac{4-2\cos\frac{2\pi j}{3n}-2\cos\frac{4\pi j}{3}}{8-2\cos\frac{2\pi j}{3n}-2\cos\frac{4\pi j}{3}}=\frac{1}{6n}\prod_{j=1}^{6n-1}[8-2\cos\frac{2\pi j}{6n}$$

$$-2\cos\frac{2\pi j}{3}]\times\prod_{j=1}^{3n-1}\frac{6-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}}{10-2\cos\frac{2\pi j}{3n}-4\cos^2\frac{2\pi j}{3}}=\frac{1}{6n}$$

$$\prod_{j=1,3\nmid j}^{6n-1}[9-2\cos\frac{2\pi j}{6n}]\times\prod_{j=1,3\mid j}^{6n-1}[6-2\cos\frac{2\pi j}{6n}]\times$$

$$\frac{\prod_{j=1,3\nmid j}^{3n-1}[5-2\cos\frac{2\pi j}{3n}]\times\prod_{j=1,3\mid j}^{3n-1}[2-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1,3\nmid j}^{3n-1}[9-2\cos\frac{2\pi j}{3n}]\times\prod_{j=1,3\mid j}^{3n-1}[6-2\cos\frac{2\pi j}{3n}]}=\frac{1}{6n}$$

$$\prod_{j=1}^{6n-1}[9-2\cos\frac{2\pi j}{6n}]\times\prod_{j=1}^{6n-1}\frac{(6-2\cos\frac{2\pi j}{6n})}{(9-2\cos\frac{2\pi j}{6n})}\times$$

$$\frac{\prod_{j=1}^{3n-1}[5-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1}^{3n-1}[9-2\cos\frac{2\pi j}{3n}]}\times\frac{\prod_{j=1}^{3n-1}\frac{(2-2\cos\frac{2\pi j}{3n})}{(5-2\cos\frac{2\pi j}{3n})}}{\prod_{j=1}^{3n-1}\frac{(6-2\cos\frac{2\pi j}{3n})}{(9-2\cos\frac{2\pi j}{3n})}}$$

Put j = 6 j' in the second and fourth terms for some integer j', we get:

$$=\frac{1}{6n}\prod_{j=1}^{6n-1}[9-2\cos\frac{2\pi j}{6n}]\times\prod_{j=1}^{2n-1}\frac{(6-2\cos\frac{2\pi j}{2n})}{(9-2\cos\frac{2\pi j}{2n})}\times$$

$$\frac{\prod_{j=1}^{3n-1}[5-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1}^{3n-1}[9-2\cos\frac{2\pi j}{3n}]}\times\frac{\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{5-2\cos\frac{2\pi j}{n}}}{\prod_{j=1}^{n-1}\frac{6-2\cos\frac{2\pi j}{n}}{9-2\cos\frac{2\pi j}{n}}}$$

$$=\frac{n}{6}\frac{U^2_{6n-1}(\sqrt{\frac{11}{4}})\cdot U^2_{2n-1}(\sqrt{2})\cdot U^2_{n-1}(\sqrt{\frac{11}{4}})\cdot U^2_{3n-1}(\sqrt{\frac{7}{4}})}{U^2_{3n-1}(\sqrt{\frac{11}{4}})\cdot U^2_{2n-1}(\sqrt{\frac{11}{4}})\cdot U^2_{n-1}(\sqrt{2})\cdot U^2_{n-1}(\sqrt{\frac{7}{4}})}$$

$$=\frac{n}{6}[(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}})^{2n}+(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}})^{2n}-1]^2\times[(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}})^{2n}$$

$$+(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}})^{2n}+1]^2\times[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n]^2$$

Theorem 7:

$$\tau(C_{6n}(1,n,2n,3n))=\frac{n}{6}[(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}})^{4n}+(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})^{4n}$$

$$+1]^2\times[(\sqrt{3}+\sqrt{2})^n+(\sqrt{3}-\sqrt{2})^n]^2\times[(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}})^{2n}+$$

$$(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}})^{2n}+1]^{-2}\times[(\sqrt{3}+\sqrt{2})^{2n}+(\sqrt{3}-\sqrt{2})^{2n}+1]^2$$

Proof: Let $\varepsilon=e^{2\pi i/6n}$. By Lemma 2.1 we have:

$$\tau(C_{6n}(1,n,2n,3n))=\frac{1}{6n}\prod_{j=1}^{6n-1}[8-\varepsilon^{-j}-\varepsilon^{-nj}-\varepsilon^{-2nj}$$

$$-\varepsilon^{-3nj}-\varepsilon^{j}-\varepsilon^{nj}-\varepsilon^{2nj}-\varepsilon^{3nj}]=\frac{1}{6n}\prod_{j=1}^{6n-1}[8-2\cos\frac{2\pi j}{6n}$$

$$-2\cos\frac{2\pi jn}{6n}-2\cos\frac{4\pi jn}{6n}-2\cos\frac{6\pi jn}{6n}]=\frac{1}{6n}$$

$$\prod_{j=1}^{6n-1}[8-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}-2\cos\frac{2\pi j}{3}-2\cos\pi j]$$

$$=\frac{1}{6n}\prod_{j=1,2\nmid j}^{6n-1}[10-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}-2\cos\frac{2\pi j}{3}]\times$$

$$\prod_{j=1,2|j}^{6n-1}[6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}-2\cos\frac{2\pi j}{3}]=\frac{1}{6n}$$

$$\prod_{j=1}^{6n-1}[10-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}-2\cos\frac{2\pi j}{3}]\times$$

$$\prod_{j=1}^{6n-1}\frac{6-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}-2\cos\frac{2\pi j}{3}}{10-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}-2\cos\frac{2\pi j}{3}}$$

Put j = 2 j′ in the second term for some integer j′, we get:

$$\tau(C_{6n}(1,n,2n,3n))=\frac{1}{6n}\prod_{j=1}^{6n-1}[10-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}-2\cos\frac{2\pi j}{3}]$$

$$\times\prod_{j=1}^{3n-1}\frac{8-2\cos\frac{2\pi j}{3n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}{12-2\cos\frac{2\pi j}{6n}-2\cos\frac{2\pi j}{3}-4\cos^2\frac{2\pi j}{3}}$$

$$=\frac{1}{6n}\prod_{j=1,3\nmid j}^{6n-1}[9-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}]$$

$$\times\prod_{j=1,3|j}^{6n-1}[8-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}]$$

$$\times\frac{\prod_{j=1,3\nmid j}^{3n-1}(8-2\cos\frac{2\pi j}{3n})\prod_{j=1,3|j}^{3n-1}(2-2\cos\frac{2\pi j}{3n})}{\prod_{j=1,3\nmid j}^{3n-1}(12-2\cos\frac{2\pi j}{3n})\prod_{j=1,3|j}^{3n-1}(6-2\cos\frac{2\pi j}{3n})}$$

$$=\frac{1}{6n}\prod_{j=1}^{6n-1}[9-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}]\times\prod_{j=1}^{6n-1}\frac{8-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}}{9-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}}$$

$$\times\frac{\prod_{j=1}^{3n-1}[8-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1}^{3n-1}[12-2\cos\frac{2\pi j}{3n}]}\times\frac{\prod_{j=1}^{3n-1}\frac{2-2\cos\frac{2\pi j}{3n}}{8-2\cos\frac{2\pi j}{3n}}}{\prod_{j=1}^{3n-1}\frac{6-2\cos\frac{2\pi j}{3n}}{9-2\cos\frac{2\pi j}{3n}}}$$

Put j = 3 j′ in the second and fourth terms for some integer j′, we get:

$$\tau(C_{6n}(1,n,2n,3n))=\frac{1}{6n}\prod_{j=1}^{6n-1}[9-2\cos\frac{2\pi j}{6n}-2\cos\frac{\pi j}{3}]\times$$

$$\times\prod_{j=1}^{2n-1}\frac{8-2\cos\frac{2\pi j}{2n}-2\cos\pi j}{9-2\cos\frac{2\pi j}{2n}-2\cos\pi j}\frac{\prod_{j=1}^{3n-1}[8-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1}^{3n-1}[12-2\cos\frac{2\pi j}{3n}]}\times\frac{\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{8-2\cos\frac{2\pi j}{n}}}{\prod_{j=1}^{n-1}\frac{6-2\cos\frac{2\pi j}{n}}{12-2\cos\frac{2\pi j}{n}}}$$

$$=\frac{1}{6n}\prod_{j=1,2\nmid j,3\nmid j}^{6n-1}[8-2\cos\frac{2\pi j}{6n}]\times\prod_{j=1,2|j,3|j}^{6n-1}[7-2\cos\frac{2\pi j}{6n}]\times$$

$$\prod_{j=1,2\nmid j,3|j}^{6n-1}[11-2\cos\frac{2\pi j}{6n}]\times\prod_{j=1,2|j,3\nmid j}^{6n-1}[10-2\cos\frac{2\pi j}{6n}]$$

$$\times\frac{\prod_{j=1,2\nmid j}^{2n-1}[10-2\cos\frac{2\pi j}{2n}]\prod_{j=1,2|j}^{2n-1}[6-2\cos\frac{2\pi j}{2n}]}{\prod_{j=1,2\nmid j}^{2n-1}[11-2\cos\frac{2\pi j}{2n}]\prod_{j=1,2|j}^{2n-1}[7-2\cos\frac{2\pi j}{2n}]}$$

$$\times\frac{\prod_{j=1}^{3n-1}[8-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1}^{3n-1}[12-2\cos\frac{2\pi j}{3n}]}\times\frac{\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{8-2\cos\frac{2\pi j}{n}}}{\prod_{j=1}^{n-1}\frac{6-2\cos\frac{2\pi j}{n}}{12-2\cos\frac{2\pi j}{n}}}=\frac{1}{6n}\prod_{j=1}^{6n-1}[8-2\cos\frac{2\pi j}{6n}]$$

$$\times\prod_{j=1}^{6n-1}\frac{(8-2\cos\frac{2\pi j}{6n})(7-2\cos\frac{2\pi j}{6n})}{(11-2\cos\frac{2\pi j}{6n})(10-2\cos\frac{2\pi j}{6n})}]\times\prod_{j=1}^{6n-1}\frac{(10-2\cos\frac{2\pi j}{6n})}{(8-2\cos\frac{2\pi j}{6n})}$$

$$\times\prod_{j=1}^{6n-1}\frac{(11-2\cos\frac{2\pi j}{6n})}{(8-2\cos\frac{2\pi j}{6n})}\times\frac{\prod_{j=1}^{2n-1}[10-2\cos\frac{2\pi j}{2n}]}{\prod_{j=1}^{2n-1}[11-2\cos\frac{2\pi j}{2n}]}\times\frac{\prod_{j=1}^{2n-1}\frac{6-2\cos\frac{2\pi j}{2n}}{10-2\cos\frac{2\pi j}{2n}}}{\prod_{j=1}^{2n-1}\frac{7-2\cos\frac{2\pi j}{2n}}{11-2\cos\frac{2\pi j}{2n}}}$$

$$\times\frac{\prod_{j=1}^{3n-1}[8-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1}^{3n-1}[12-2\cos\frac{2\pi j}{3n}]}\times\frac{\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{8-2\cos\frac{2\pi j}{n}}}{\prod_{j=1}^{n-1}\frac{6-2\cos\frac{2\pi j}{n}}{12-2\cos\frac{2\pi j}{n}}}$$

Put j = 6 j' in the second term, j = 2 j' in the third term, j = 3 j' in the fourth term and j = 2 j' in the sixth term for some integer j', we get:

$$\tau(C_{6n}(1,n,2n,3n))=\frac{1}{6n}\prod_{j=1}^{6n-1}[8-2\cos\frac{2\pi j}{6n}]\times$$

$$\prod_{j=1}^{n-1}\frac{(8-2\cos\frac{2\pi j}{n})(7-2\cos\frac{2\pi j}{n})}{(11-2\cos\frac{2\pi j}{n})(10-2\cos\frac{2\pi j}{n})}]\times\prod_{j=1}^{3n-1}\frac{(10-2\cos\frac{2\pi j}{3n})}{(8-2\cos\frac{2\pi j}{3n})}$$

$$\times\prod_{j=1}^{2n-1}\frac{(11-2\cos\frac{2\pi j}{2n})}{(8-2\cos\frac{2\pi j}{2n})}\times\frac{\prod_{j=1}^{2n-1}[10-2\cos\frac{2\pi j}{2n}]}{\prod_{j=1}^{2n-1}[11-2\cos\frac{2\pi j}{2n}]}\times\frac{\prod_{j=1}^{n-1}\frac{6-2\cos\frac{2\pi j}{n}}{10-2\cos\frac{2\pi j}{n}}}{\prod_{j=1}^{n-1}\frac{7-2\cos\frac{2\pi j}{n}}{11-2\cos\frac{2\pi j}{n}}}$$

$$\times\frac{\prod_{j=1}^{3n-1}[8-2\cos\frac{2\pi j}{3n}]}{\prod_{j=1}^{3n-1}[12-2\cos\frac{2\pi j}{3n}]}\times\frac{\prod_{j=1}^{n-1}\frac{2-2\cos\frac{2\pi j}{n}}{8-2\cos\frac{2\pi j}{n}}}{\prod_{j=1}^{n-1}\frac{6-2\cos\frac{2\pi j}{n}}{12-2\cos\frac{2\pi j}{n}}}$$

$$=\frac{n}{6}\frac{U^2_{6n-1}(\sqrt{\frac{5}{2}})\cdot U^2_{n-1}(\sqrt{\frac{7}{2}})\cdot U^2_{3n-1}(\sqrt{3})\cdot U^2_{2n-1}(\sqrt{3})}{U^2_{2n-1}(\sqrt{\frac{5}{2}})\cdot U^2_{3n-1}(\sqrt{\frac{7}{2}})\cdot U^2_{n-1}(\sqrt{3})\cdot U^2_{n-1}(\sqrt{3})}$$

$$=\frac{n}{6}[(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}})^{4n}+(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})^{4n}+1]^2\times[(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}})^{2n}$$
$$+(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}})^{2n}+1]^{-2}\times[(\sqrt{3}+\sqrt{2})^{2n}+(\sqrt{3}-\sqrt{2})^{2n}+1]^2\times$$
$$[(\sqrt{3}+\sqrt{2})^{n}+(\sqrt{3}-\sqrt{2})^{n}]^2$$

Now we conclude that a few more applications (proofs omitted).

Theorem 8:

$$\tau(C_{12n}(1,2n,3n))=\frac{n}{12}[(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}})^{4n}+(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}})^{4n}-1]^2$$
$$\times[(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}})^{2n}+(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}})^{2n}+1]^2$$
$$\times[(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}})^{2n}+(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})^{2n}]^2$$
$$\times[(\sqrt{2}+1)^{n}+(\sqrt{2}-1)^{n}]^2$$
$$\times[(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}})^{2n}+(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}})^{2n}-1]^2$$

Theorem 9:

$$\tau(C_{12n}(1,3n,6n))=\frac{3n}{4}[(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}})^{6n}+(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})^{6n}]^2$$
$$\times[(\sqrt{2}+1)^{3n}+(\sqrt{2}-1)^{3n}]^2$$

Theorem 10:

$$\tau(C_{12n}(1,3n,4n))=\frac{n}{12}[(\sqrt{\frac{9}{4}}+\sqrt{\frac{5}{4}})^{4n}+(\sqrt{\frac{9}{4}}-\sqrt{\frac{5}{4}})^{4n}-1]^2$$
$$\times[(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}})^{2n}+(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}})^{2n}+1]^2$$
$$\times\ [(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}})^{2n}+(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}})^{2n}]^2$$
$$\times[(\sqrt{2}+1)^{n}+(\sqrt{2}-1)^{n}]^2$$
$$\times[(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}})^{2n}+(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}})^{2n}-1]^2$$

Theorem 11:

$$\tau(C_{12n}(1,2n,3n,4n))=\frac{n}{12}[(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}})^{6n}+(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})^{6n}]^2$$
$$\times[(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}})^{2n}+(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})^{2n}+1]^2\times[(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}})^{2n}+$$
$$(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}})^{2n}-1]^2\times[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n]^2$$

Theorem 12:

$$\tau(C_{12n}(1,2n,3n,6n))=\frac{n}{12}[(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}})^{4n}+(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}})^{4n}-1]^2$$
$$\times[(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}})^{2n}+(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}})^{2n}+1]^{-2}$$
$$\times[(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}})^{4n}+(\sqrt{\frac{13}{4}}-\sqrt{\frac{9}{4}})^{4n}+1]^2$$
$$\times[(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}})^{2n}+(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}})^{2n}]^2$$
$$\times[(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}})^{2n}+(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}})^{2n}+1]^2$$
$$\times[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n]^2$$

Theorem 13:

$$\tau(C_{12n}(1,3n,4n,6n))=\frac{n}{12}[(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}})^{8n}+(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}})^{8n}+1]^2$$
$$\times[(\sqrt{\frac{11}{4}}+\sqrt{\frac{7}{4}})^{2n}+(\sqrt{\frac{11}{4}}-\sqrt{\frac{7}{4}})^{2n}-1]^2$$
$$\times[(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}})^{2n}+(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})^{2n}]^2$$
$$\times[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n]^2$$
$$\times[(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}})^{2n}+(\sqrt{\frac{7}{4}}-\sqrt{\frac{3}{4}})^{2n}+1]^2$$
$$\times[(\sqrt{\frac{13}{4}}+\sqrt{\frac{9}{4}})^{4n}+(\sqrt{\frac{13}{4}}-\sqrt{\frac{9}{4}})^{4n}+1]^{-2}$$

Theorem 14:

$$\tau(C_{12n}(1,2n,3n,4n,6n))=\frac{n}{12}[(\sqrt{3}+\sqrt{2})^{4n}+(\sqrt{3}-\sqrt{2})^{4n}-1]^2$$
$$\times[(\sqrt{2}+1)^n+(\sqrt{2}-1)^n]^2\times[(\sqrt{\frac{5}{2}}+\sqrt{\frac{3}{2}})^{2n}+(\sqrt{\frac{5}{2}}-\sqrt{\frac{3}{2}})^{2n}+1]^2$$
$$\times[(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}})^{2n}+(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}})^{2n}-1]^2$$
$$\times[(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}})^{2n}+(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}})^{2n}]^2$$
$$\times[(\sqrt{\frac{7}{2}}+\sqrt{\frac{5}{2}})^{4n}+(\sqrt{\frac{7}{2}}-\sqrt{\frac{5}{2}})^{4n}+1]^2$$
$$\times[(2+\sqrt{3})^{4n}+(2-\sqrt{3})^{4n}+1]^{-2}$$

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