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ORDER STATISTICS FROM INDEPENDENT NON-IDENTICALLY DISTRIBUTED DISCRETE RAYLEIGH DISTRIBUTION

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ABSTRACT

This study deals with the subject of Order Statistics (OS) and its moments for independent non-identically distributed discrete random variables rv's drawn from one parameter discrete Rayleigh distribution or dRayleigh (θ). Distributions of single order statistics and their moments for dRayleigh (θ) are studied. The k^{th} moments of single os and the mean of the largest and smallest os are obtained for n = 3.

Keywords: Order Statistics, Discrete Random Variable, Discrete Rayleigh Distribution, Permanents

1. INTRODUCTION

The subject of order statistics for Independent Identically Distributed (IID) discrete random variates is referred to the work published by (Raider, 1951; Abdel-Aty, 1954; Siotani, 1956; Kabe, 1969; Arnold *et al.*, 1992; Nagaraga, 1992). Not much attention has been paid to the case when assuming that the random variables are discrete and Independent Non-Identically Distributed (INID). Basic distribution theory of order statistics for this case was first discussed by (Gungor *et al.*, 2009). The appearance of new discrete distributions in the literature such as: discrete gamma distribution (Chakraborty, 2012; Alhazzani, 2012), discrete burr and pareto distribution (Krishna and Pundir, 2009), discrete Weibull distribution (Nakagawa and Osaki, 1975; Stein and Dattero, 1984; Khan *et al.*, 1989), discrete normal distribution (Roy, 2003) and discrete Rayleigh distribution (Roy, 2004) opens the door for lots of scientific papers in order statistics for both iid and inid cases.

Let $X_1, X_2, \ldots, X_n$ be independent non-identically distributed (inid) random variables having adiscrete cumulative distribution function cdf $F_i(x)$ and probability mass function pmf $f_i(x)$ respectively.

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denotes the order statistics obtained by arranging the n of Xi's in increasing order of magnitude and $F_{r:n}(x)$ (r = 1,2,..,n) be the cdf of the r^{th} order statistics $X_{r:n}$.

In this study we will study os from discrete Rayleigh distribution dRayleigh (θ) when the X_i's are inid random variables.

1.1. Discrete Rayleigh Distribution

The discrete Rayleighdistribution was obtained and studied by (Roy, 2004), by using the general approach of discretizing a continuous distribution. Theprobability mass function (pmf) and the cumulative distribution function (cdf) of the dRayleigh (θ)are given by Equation 1 and 2:

$$p(x) = \Pr(X = x) = \theta^{x^2} - \theta^{(x+1)^2}, X = 0,1,2,\ldots, 0 < \theta < 1 \quad (1)$$

$$P(x) = \Pr(X \le x) = 1 - \theta^{x^2} \quad (2)$$

Let $X_1, X_2, \ldots, X_n$ be Independent Non-Identically Distributed (INID) random variables having dRayleighdistributions Equation 3:

$$p_i(x) = \Pr(X = x) = \theta_i^{x^2} - \theta_i^{(x+1)^2}, X = 0,1,2,\ldots, 0 < \theta_i < 1 \quad (3)$$

1.2. Distribution of single os from inid dRayleigh (θ) rv's

The general form of the pmf of r^{th} os; $X_{r:n}$ from inid discrete distribution is given by (Gungor *et al.*, 2009) Equation 4:

$$p_{r:n}(x) = P_r(X_{r:n} = x) = \sum_{k=0}^{r-1}\sum_{m=0}^{n-r}\frac{\sum_P}{(r-k-1)!(k+m+1)!(n-r-m)!} \Pi_{a=1}^{r-k-1} P_{i_a}(x-1)\Pi_{b=r-k}^{k+m+1} p_{i_b}(x)\Pi_{c=k+m+2}^{n}(1-P_{i_c}(x)) \quad (4)$$

where, i = 1,2,...,n and Σ_P denotes the summation over all n! permutations $(i_1,i_2,\ldots\ldots i_n)$ of $(1,2,3,\ldots,n)$.

The pmf of r^{th}os; $X_{r:n}$ from iniddRayleigh (θ) distribution is given by Equation 5:

$$p_{r:n}(x) = \sum_{k=0}^{r-1}\sum_{m=0}^{n-r}\frac{\sum_P}{(r-k-1)!(k+m+1)!(n-r-m)!} \prod_{a=1}^{r-k-1}(1-\theta_{i_a}^{(x-1)^2})\prod_{b=r-k}^{k+m+1}(\theta_{i_b}^{x^2}-\theta_{i_b}^{(x+1)^2})\prod_{c=k+m+2}^{n}\left(\theta_{i_c}^{x^2}\right) \quad (5)$$

Using permanent notation the above pmf can be written as (Vaughan and Venables, 1972) Equation 6:

$$p_{r:n}(x) = \sum_{k=0}^{r-1}\sum_{m=0}^{n-r}\frac{\sum_P}{(r-k-1)!(k+m+1)!(n-r-m)!} \operatorname{per}\left[\underbrace{1-\theta^{(x-1)^2}}_{r-k-1}\ \underbrace{\theta^{x^2}-\theta^{(x+1)^2}}_{k+m+1}\ \underbrace{\theta^{x^2}}_{n-r-m}\right] \quad (6)$$

when, k = m = 0 i.e., the chance of ties is zero and all the observations are distinct, an easier form of the pmf of $X_{r:n}$ is now obtained Equation 7 and 8:

$$p_{r:n}(x) = \frac{\sum_P}{(r-1)!(n-r)!}\prod_{a=1}^{r-1}(1-\theta_{i_a}^{(x-1)^2}) [\theta_{i_r}^{x^2}-\theta_{i_r}^{(x+1)^2}]\prod_{c=r+1}^{n}\left(\theta_{i_e}^{x^2}\right) \quad (7)$$

$$P_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} \operatorname{per}\left[\underbrace{1-\theta^{(x-1)^2}}_{r-1}\ \underbrace{\theta^{x^2}-\theta^{(x-1)^2}}_{1}\ \underbrace{\theta^{x^2}}_{n-r}\right] \quad (8)$$

1.3. Distribution of the Smallest and Largest os Following inid dRayleigh (θ) rv's

The pmf of $X_{1:n}$ and $X_{n:n}$ from inid dRayleigh (θ) are given by Equation 9 and 10:

$$p_{1:n}(x) = \frac{1}{(n-1)!}\operatorname{per}\left[\underbrace{\theta^{x^2}-\theta^{(x+1)^2}}_{1}\ \underbrace{\theta^{x^2}}_{n-1}\right] \quad (9)$$

$$p_{n:n}(x) = \frac{1}{(n-1!)}\operatorname{per}\left[\underbrace{1-\theta^{(x-1)^2}}_{n-1}\ \underbrace{\theta^{x^2}-\theta^{(x-1)^2}}_{1}\right] \quad (10)$$

Example 1

For n = 3 the pmf of $X_{1:3}$ and $X_{3:3}$ from iniddRayleigh (θ) are given by Equation 11:

$$p_{1:3}(x) = \frac{1}{2!}\operatorname{per}\left[\underbrace{\theta^{x^2}-\theta^{(x+1)^2}}_{1}\ \underbrace{\theta^{x^2}}_{2}\right] = \frac{1}{2}\operatorname{per}\begin{bmatrix}\theta_1^{x^2}-\theta_1^{(x+1)^2} & \theta_1^{x^2} & \theta_1^{x^2}\\ \theta_2^{x^2}-\theta_2^{(x+1)^2} & \theta_2^{x^2} & \theta_2^{x^2}\\ \theta_3^{x^2}-\theta_3^{(x+1)^2} & \theta_3^{x^2} & \theta_3^{x^2}\end{bmatrix} = (\theta_1\theta_2\theta_3)^{x^2}(3-\theta_1^{1+2x}-\theta_2^{1+2x}-\theta_3^{1+2x})$$

$$p_{3:3}(x) = \frac{1}{2}\operatorname{per}\left[\underbrace{1-\theta^{(x-1)^2}}_{2}\ \underbrace{\theta^{x^2}-\theta^{(x-1)^2}}_{1}\right] = \frac{1}{2}\operatorname{per}\begin{bmatrix}1-\theta_1^{(x-1)^2} & 1-\theta_1^{(x-1)^2} & \theta_1^{x^2}-\theta_1^{(x-1)^2}\\ 1-\theta_2^{(x-1)^2} & 1-\theta_2^{(x-1)^2} & \theta_2^{x^2}-\theta_2^{(x-1)^2}\\ 1-\theta_3^{(x-1)^2} & 1-\theta_3^{(x-1)^2} & \theta_3^{x^2}-\theta_3^{(x-1)^2}\end{bmatrix} \quad (11)$$

$$= (\theta_1^{x^2}-\theta_1^{(1+x)^2})(-1+\theta_2^{(-1+x)^2})(-1+\theta_3^{(-1+x)^2}) + (-1+\theta_1^{(-1+x)^2})(\theta_2^{x^2}-\theta_2^{(1+x)^2})(-1+\theta_3^{(-1+x)^2}) + (-1+\theta_1^{(-1+x)^2})(-1+\theta_2^{(-1+x)^2})(\theta_3^{x^2}-\theta_3^{(1+x)^2})$$

The cdf of $X_{r:n}$ from iniddRayleigh (θ)is given by Equation 12:

$$p_{r:n}(x) = \sum_{j=r}^{n}\sum_{Pj}\prod_{a=1}^{j}[1-\theta_{i_a}^{x^2}]\prod_{b=j+1}^{n}[\theta_{i_b}^{x^2}] \quad (12)$$

where, Σ_P denotes the summation over allpermutations $(i_1,i_2,\ldots,i_n)$ of $(1,2,3,\ldots,n)$ for which $i_1<i_2<\ldots<i_j$and $i_{j+1}<i_{j+2}<\ldots<i_n$ Equation 13:

$$p_{r:n}(x) = \sum_{j=r}^{n}\frac{1}{j!(n-j)!}\operatorname{per}\left[\underbrace{1-\theta^{x^2}}_{j}\ \underbrace{\theta^{x^2}}_{n-j}\right] \quad (13)$$

Then the cdf of $X_{1:n}$ and $X_{n:n}$ are Equation 14-16:

$$p_{1:n}(x) = \sum_{j=1}^{n}\sum_{pj}\prod_{a=1}^{j}\left(1-\theta_{i_a}^{x^2}\right)\prod_{b=j+1}^{n}\left(\theta_{i_b}^{x^2}\right) \quad (14)$$

Table 1. The mean of the smallet os from inid dRayleigh (θ)

1 -------------------------- 2	3	0.1	0.2	0.3	0.4	0.5
0.1	0.1	1.001	1.025	1.05001	1.07503	1.10010
	0.2	1.002	1.002	1.00300	1.00400	1.00500
	0.3	1.003	1.004	1.00600	1.00800	1.01000
	0.4	1.004	1.006	1.00900	1.01200	1.01500
	0.5	1.005	1.008	1.01200	1.01600	1.02000
0.2	0.1	1.002	1.010	1.01500	1.02000	1.02500
	0.2	1.004	1.004	1.00600	1.00800	1.01000
	0.3	1.006	1.008	1.01200	1.01600	1.02000
	0.4	1.008	1.012	1.01800	1.02400	1.03000
	0.5	1.010	1.016	1.02400	1.03200	1.04000
0.3	0.1	1.010	1.020	1.03000	1.04000	1.05001
	0.2	1.003	1.006	1.00900	1.01200	1.01500
	0.3	1.006	1.012	1.01800	1.02400	1.03000
	0.4	1.009	1.018	1.02700	1.03600	1.04500
	0.5	1.012	1.024	1.03600	1.04801	1.06001
0.4	0.1	1.015	1.030	1.04500	1.06001	1.02000
	0.2	1.004	1.008	1.01200	1.01600	1.04000
	0.3	1.008	1.016	1.02400	1.03200	1.06001
	0.4	1.012	1.024	1.03600	1.04801	1.08004
	0.5	1.016	1.032	1.04801	1.06402	1.10010
0.5	0.1	1.020	1.040	1.06001	1.08004	1.02500
	0.2	1.005	1.010	1.01500	1.02000	1.05001
	0.3	1.010	1.020	1.03000	1.04000	1.07503
	0.4	1.015	1.030	1.04500	1.06001	1.10010
	0.5	1.020	1.040	1.06001	1.08004	1.12524

Table 2. The mean of the largest os from inid dRayleigh (θ)

1 ----------------------- 2	2	0.1	0.2	0.3	0.4	0.5
0.1	0.1	1.27130	1.35380	1.44130	1.53979	1.65769
	0.2	1.35380	1.42730	1.50579	1.59526	1.70409
	0.3	1.44130	1.50579	1.57523	1.65559	1.75519
	0.4	1.53979	1.59526	1.65559	1.72664	1.81659
	0.5	1.65769	1.70409	1.75519	1.81659	1.89618
0.2	0.1	1.35380	1.42730	1.50579	1.59526	1.70409
	0.2	1.42730	1.49279	1.56327	1.64472	1.74550
	0.3	1.50579	1.56327	1.62571	1.69904	1.79158
	0.4	1.59526	1.64472	1.69904	1.76406	1.84796
	0.5	1.70409	1.74550	1.79158	1.84796	1.92250
0.3	0.1	1.44130	1.50579	1.57523	1.65559	1.75519
	0.2	1.50579	1.56327	1.62571	1.69904	1.79158
	0.3	1.57523	1.62571	1.68110	1.74732	1.83263
	0.4	1.65559	1.69904	1.74732	1.80624	1.88390
	0.5	1.75519	1.79158	1.83263	1.88390	1.95321
0.4	0.1	1.53979	1.59526	1.65559	1.72664	1.81659
	0.2	1.59526	1.64472	1.69904	1.76406	1.84796
	0.3	1.65559	1.69904	1.74732	1.80624	1.88390
	0.4	1.72664	1.76406	1.80624	1.85885	1.92989
	0.5	1.81659	1.84796	1.88390	1.92989	1.99359
0.5	0.1	1.65769	1.70409	1.75519	1.81659	1.89618
	0.2	1.70409	1.74550	1.79158	1.84796	1.92250
	0.3	1.75519	1.79158	1.83263	1.88390	1.95321
	0.4	1.81659	1.84796	1.88390	1.92989	1.99359
	0.5	1.89618	1.92250	1.95321	1.99359	2.05103

$$P_{1:n}(x)=\sum_{j=1}^{n}\frac{1}{j!(n-j)!}\operatorname{per}\left[\underbrace{1-\theta^{x^2}}_{j}\ \underbrace{\theta^{x^2}}_{n-j}\right] \quad (15)$$

$$P_{n:n}(x)=\prod_{a=1}^{n}\left(1-\theta_{a_i}^{x^2}\right)=\frac{1}{n!}\operatorname{per}\left[\underbrace{1-\theta^{x^2}}_{n}\right] \quad (16)$$

Example 2

For n = 3 the cdf of $X_{1:3}$ from iniddRayleigh (θ) is given by Equation 17:

$$\begin{aligned}
P_{1:3}(x)&=\sum_{j=1}^{3}\frac{1}{j!(3-j)!}\operatorname{per}\left[\underbrace{1-\theta^{x^2}}_{j}\ \underbrace{\theta^{x^2}}_{3-j}\right]\\
&=\frac{1}{2}\operatorname{per}\left[\underbrace{1-\theta^{x^2}}_{1}\ \underbrace{\theta^{x^2}}_{2}\right]+\frac{1}{2}\operatorname{per}\left[\underbrace{1-\theta^{x^2}}_{2}\ \underbrace{\theta^{x^2}}_{1}\right]+\frac{1}{3!}\operatorname{per}\left[\underbrace{1-\theta^{x^2}}_{3}\right]\\
&=\frac{1}{2}\operatorname{per}\begin{bmatrix}1-\theta_1^{x^2} & \theta_1^{x^2} & \theta_1^{x^2}\\ 1-\theta_2^{x^2} & \theta_2^{x^2} & \theta_2^{x^2}\\ 1-\theta_3^{x^2} & \theta_3^{x^2} & \theta_3^{x^2}\end{bmatrix}
+\frac{1}{2}\operatorname{per}\begin{bmatrix}1-\theta_1^{x^2} & 1-\theta_1^{x^2} & \theta_1^{x^2}\\ 1-\theta_2^{x^2} & 1-\theta_2^{x^2} & \theta_2^{x^2}\\ 1-\theta_3^{x^2} & 1-\theta_3^{x^2} & \theta_3^{x^2}\end{bmatrix}\\
&\quad+\frac{1}{6}\operatorname{per}\begin{bmatrix}1-\theta_1^{x^2} & 1-\theta_1^{x^2} & 1-\theta_1^{x^2}\\ 1-\theta_2^{x^2} & 1-\theta_2^{x^2} & 1-\theta_2^{x^2}\\ 1-\theta_3^{x^2} & 1-\theta_3^{x^2} & 1-\theta_3^{x^2}\end{bmatrix}\\
&=1-\theta_1^{x^2}\theta_2^{x^2}\theta_3^{x^2}
\end{aligned} \quad (17)$$

For n = 3 the cdf of $X_{3:3}$ from iniddRayleigh (θ) is given by Equation 18:

$$\begin{aligned}
P_{3:3}(x)&=\frac{1}{3!}\operatorname{per}\left[\underbrace{1-\theta^{x^2}}_{3}\right]
=\frac{1}{6}\operatorname{per}\begin{bmatrix}1-\theta_1^{x^2} & 1-\theta_1^{x^2} & 1-\theta_1^{x^2}\\ 1-\theta_2^{x^2} & 1-\theta_2^{x^2} & 1-\theta_2^{x^2}\\ 1-\theta_3^{x^2} & 1-\theta_3^{x^2} & 1-\theta_3^{x^2}\end{bmatrix}\\
&=-\left(-1+\theta_1^{x^2}\right)\left(-1+\theta_2^{x^2}\right)\left(-1+\theta_3^{x^2}\right)
\end{aligned} \quad (18)$$

1.4. Moments of os from Inid dRayleigh (θ) rv's

The k^{th} moment of $X_{r:n}$ from Inid dRayleigh (θ) is given by Equation 19:

$$\mu_{r:n}^{(k)}=M^k-\sum_{x=0}^{M-1}\sum_{j=r}^{n}\sum_{P_j}\prod_{a=1}^{j}\left[1-\theta_{i_a}^{x^2}\right]\prod_{b=j+1}^{n}\left[\theta_{i_b}^{x^2}\right]\left[(x+1)^k-x^k\right] \quad (19)$$

The mean of the r^{th} os is Equation 20:

$$\mu_{r:n}=\sum_{x=0}^{M-1}\sum_{j=r}^{n}\sum_{P_j}\prod_{a=1}^{j}\left[1-\theta_{i_a}^{x^2}\right]\prod_{b=j+1}^{n}\left[\theta_{i_b}^{x^2}\right] \quad (20)$$

and the mean for the smallest and largest os from IniddRayleigh (θ) are given by Equation 21 and 22:

$$\mu_{1:n}=\sum_{x=0}^{M-1}\prod_{i=1}^{n}\left[\theta_i^{x^2}\right] \quad (21)$$

$$\mu_{n:n}=\sum_{x=0}^{M-1}\left(1-\prod_{i=1}^{n}\left[1-\theta_i^{x^2}\right]\right) \quad (22)$$

Example 3

For M = 3, n = 3, the mean of the smallest and largest os are then Equation 23 and 24:

$$\mu_{1:3}=\sum_{x=0}^{2}\prod_{i=1}^{3}\left[\theta_i^{x^2}\right] \quad (23)$$

$$\mu_{3:3}=\sum_{x=0}^{2}\left(1-\prod_{i=1}^{3}\left[1-\theta_i^{x^2}\right]\right) \quad (24)$$

Calculations of the mean of the smallest and largest os from dRayleigh (θ) are given in **Table 1 and 2** for several values of 1, 2, 3.

2. CONCLUSION

Discrete Rayleigh distribution under contaminations (INID) is introduced. Exact forms of the pmf and cdf of order statistics and single moment are derived. Further studies are suggested for other discrete distributions.

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