Journal of Mathematics and Statistics

Original Research Paper

A New Inverse Weibull Distribution: Properties and Applications

[1]Mona Mustafa Elbiely and [2]Haitham M. Yousof

[1]Department of Statistics, Mathematics and Insurance, Damanhur University, Egypt
[2]Department of Statistics, Mathematics and Insurance, Benha University, Benha, Egypt

Article history
Received: 22-01-2019
Revised: 18-02-2019
Accepted: 18-03-2019

Corresponding Author:
Mona Mustafa Elbiely
Department of Statistics,
Mathematics and Insurance,
Damanhur University, Egypt
Email: elbielym@gmail.com

Abstract: In this work, we introduce a new flexible extension of the Inverse Weibull distribution. Some important mathematical properties of the proposed model are derived along with a numerical analysis of mean, variance, skewness and kurtosis measures of the proposed model. The performance of the maximum likelihood method is assesses via a comprehensive simulation studies in terms of mean squared errors. The new model is better than some other important competitive extensions of the Inverse Weibull in modeling the breaking stress data, the glass fibers data and the relief time data. Some plots are given to illustrate the suitability of the new model to fit the used data sets.

Keywords: Inverse Weibull Distribution, Extreme Values, Moments, Estimation

Introduction

A Random Variable (RV) T is said to have the Inverse Weibull (IW) distribution (Gusmao *et al*., 2011) if its Probability Density Function (PDF) and Cumulative Distribution Function (CDF) are given by:

$$\pi_{(c,a,b)}(t) = cba^b t^{-(b+1)} e^{-c\left(at^{-1}\right)^b} \big|_{[x\geq 0]}, \quad (1)$$

and:

$$\prod\nolimits_{(c,a,b)}(t) = e^{-c\left(at^{-1}\right)^b} \big|_{[t\geq 0]}, \quad (2)$$

respectively, where $a > 0$ is a scale parameter and c, $b > 0$ are a shape parameters, respectively. For $c = 1$, we get the standard IW model. For $b = 2$ in (2), we get the Generalized Inverse Rayleigh model (GIR). For $a = 1$ in (2), we get the generalized Inverse Exponential model (GIEx). For $c = 1$ and $b = 2$ in (2), we get the IR model. For $c = a = 1$ we get the IEx model.

Recently, Cordeiro *et al*. (2016) proposed a new class of distributions called the generalized odd log-logistic-G (GOLL-G) family with two extra shape parameters. For an arbitrary baseline CDF $G_{\underline{\Phi}}(x)$ and $\underline{\Phi}$ the parameters vector, the CDF of the GOLL-G family is given by:

$$F_{\alpha,\theta,\underline{\Phi}}(x) = \frac{G_{\underline{\Phi}}(x)^{\alpha\theta}}{G_{\underline{\Phi}}(x)^{\alpha\theta} + \left[1 - G_{\underline{\Phi}}(x)^{\theta}\right]^{\alpha}}. \quad (3)$$

The PDF corresponding to (3) is given by:

$$f_{\alpha,\theta,\underline{\Phi}}(x) = \frac{\alpha\theta g_{\underline{\Phi}}(x) G_{\underline{\Phi}}(x)^{\alpha\theta-1}\left[1 - G_{\underline{\Phi}}(x)^{\theta}\right]^{\alpha-1}}{\left\{G_{\underline{\Phi}}(x)^{\alpha\theta} + \left[1 - G_{\underline{\Phi}}(x)^{\theta}\right]^{\alpha}\right\}^2}. \quad (4)$$

Here, we define the generalized odd log-logistic Inverse Weibull (GOLLIW) model and provide some plots of its PDF and hazard rate function (HRF) [$h_{(\alpha,\theta,c,a,b)}(x)$]. The GOLLIW CDF is given by:

$$F_{(\alpha,\theta,c,a,b)}(x) = \frac{e^{-\alpha\theta c\left(ax^{-1}\right)^b}}{e^{-\alpha\theta c\left(ax^{-1}\right)^b} + \left[1 - e^{-\theta c\left(ax^{-1}\right)^b}\right]^{\alpha}}. \quad (5)$$

The PDF corresponding to (5) is given by:

$$f_{(\alpha,\theta,c,a,b)}(x) = \frac{\alpha\theta cba^b x^{-(b+1)} e^{-\alpha\theta c\left(ax^{-1}\right)^b}\left[1 - e^{-\theta c\left(ax^{-1}\right)^b}\right]^{\alpha-1}}{\left\{e^{-\alpha\theta c\left(ax^{-1}\right)^b} + \left[1 - e^{-\theta c\left(ax^{-1}\right)^b}\right]^{\alpha}\right\}^2}. \quad (6)$$

The Hazard Rate Function (HRF) for the new model can be get from $f_{(\alpha,\theta,c,a,b)}(x)/[1-F_{(\alpha,\theta,c,a,b)}(x)]$.

Let $\tau = \inf\left\{x\,|_{\Pi_{(c,a,b)}(x)>0}\right\}$ and the asymptotics of the CDF, PDF and HRF as $x \to \tau$ are given by:

$$F_{(\alpha,\theta,c,a,b)}(x) \sim e^{-\alpha\theta c\left(ax^{-1}\right)^{b}}\Big|_{[x\to\tau]},$$

$$f_{(\alpha,\theta,c,a,b)}(x) \sim \alpha\theta cba^{b}x^{-(b+1)}e^{-\alpha\theta c\left(ax^{-1}\right)^{b}}\Big|_{[x\to\tau]}$$

and:

$$h_{(\alpha,\theta,c,a,b)}(x) \sim \alpha\theta cba^{b}x^{-(b+1)}e^{-\alpha\theta c\left(ax^{-1}\right)^{b}}\Big|_{[x\to\tau]}$$

The asymptotics of CDF and HRF as $x\to\infty$ are given by:

$$1-F_{(\alpha,\theta,c,a,b)}(x) \sim \theta^{\alpha}\left[1-e^{-c\left(ax^{-1}\right)^{b}}\right]^{\alpha}\Big|_{[x\to\infty]},$$

$$f_{(\alpha,\theta,c,a,b)}(x) \sim \alpha\theta^{\alpha}cba^{b}x^{-(b+1)}e^{-c\left(ax^{-1}\right)^{b}}\left[1-e^{-c\left(ax^{-1}\right)^{b}}\right]^{\alpha-1}\Big|_{[x\to\infty]}$$

and:

$$h_{(\alpha,\theta,c,a,b)}(x) \sim \alpha cba^{b}x^{-(b+1)}e^{-c\left(ax^{-1}\right)^{b}}\left[1-e^{-c\left(ax^{-1}\right)^{b}}\right]^{-1}\Big|_{[x\to\infty]}.$$

Table 1 provide the sub-models of the GOLLIW model. As illustrated in Table 1, the new model generalizes nineteen sub-model, ten of them are quite new.

The density of the proposed model can be right skewed and symmetric (see left panel in Fig. 1). The HRF of the proposed model can be only upside down (see right panel in Fig. 1 and 3 to 5).

For simulation of this new model, we obtain the Quantile Function (QF) of X (by inverting (5)), say $x_u = Q(u) = F^{-1}(u)$, as:

$$x_u = a\left[\sqrt[b]{\left(-c^{-1}\ln\left\{\sqrt[\alpha\theta]{\left(\frac{u}{1-u}\right)}\sqrt[\theta]{\left[\left(\frac{u}{1-u}\right)^{\frac{1}{\alpha}}+1\right]^{-1}}\right\}\right)^{-1}}\right] \quad (7)$$

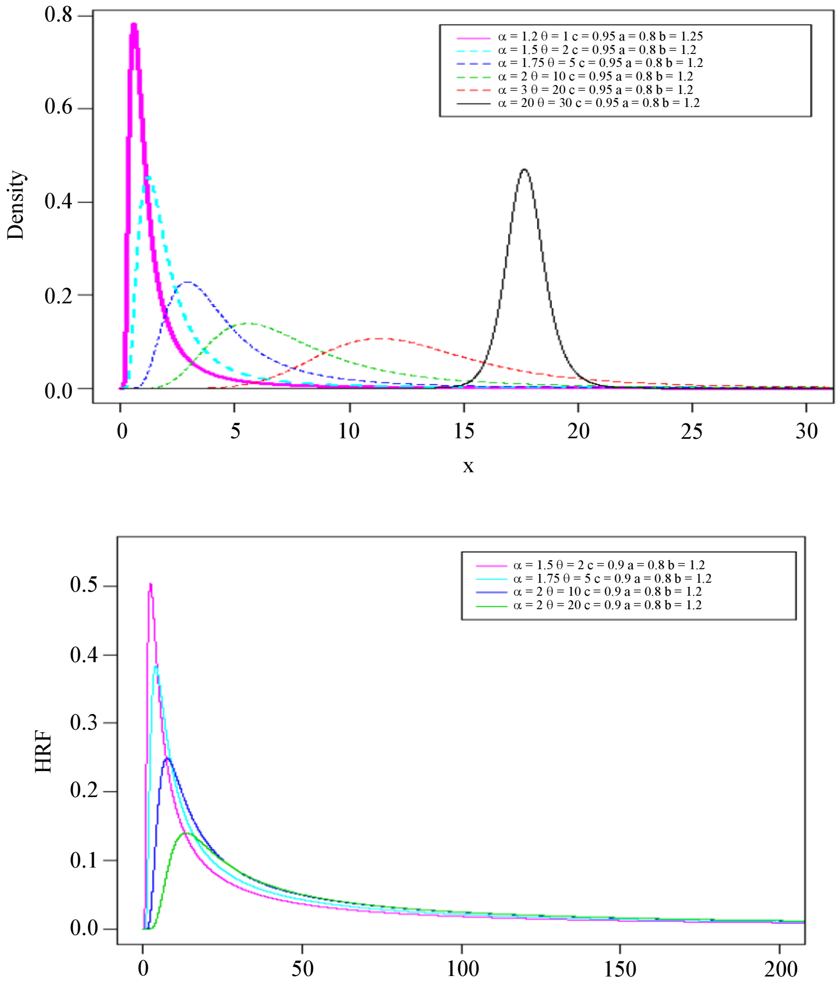

Fig. 1: Plots of the GOLLIW PDF and HRF

Table 1: Sub-models of the GOLLIW model

N	α	θ	c	a	b	Reduced model	Author
1		1				OLLGIW	New
2		1			2	OLLGIR	New
3		1			1	OLLGIEx	New
4		1	1			OLLGIW	Yousof *et al.* (2018)
5		1	1		2	OLLGIR	Yousof *et al.* (2018)
6		1	1		1	OLLGIEx	Yousof *et al.* (2018)
7			1			GOLLIW	New
8			1		2	GOLLIR	New
9			1		1	GOLLIEx	New
10	1					Proportional reversed hazard rate GIW (PRHRGIW)	New
11	1				2	PRHRGIR	New
12	1				1	PRHRGIEx	New
13	1		1			PRHRIW	New
14	1		1		2	PRHRIR	New
15	1		1		1	PRHRIEx	New
16	1	1				GIW	Nadarajah and Kotz (2003) and Gusmao *et al.* (2011)
17	1	1			2	GIR	Nadarajah and Kotz (2003) and Gusmao *et al.* (2011)
18	1	1			1	GIEx	Nadarajah and Kotz (2003) and Gusmao *et al.* (2011)
19	1	1			1	Fr	Fréchet (1927)
20	1	1	1		2	IR	Treyer (1964)
21						IEx	Keller and Kamath (1982)

We used (7) for simulating the new model (see Section 4).

Some important extensions of the IW model can be cited as: The exponentiated IW by Nadarajah and Kotz (2003) beta IW by Barreto-Souza *et al*. (2011), transmuted IW by Mahmoud and Mandouh (2013), Marshall- Olkin IW by Krishna *et al*. (2013), transmuted Marshall-Olkin IW by Afify *et al*. (2015), beta exponential IW by Mead *et al*. (2017), among others.

Mathematical Properties

Useful Representations

Following Cordeiro *et al*. (2016) and after some algebra, the PDF (6) of X can be expressed as:

$$f_{(\alpha,\theta,c,a,b)}(x)=\sum_{\zeta=0}^{\infty} b_{\zeta}\pi_{((1+\zeta)c,a,b)}(x), \tag{8}$$

where:

$$b_{\zeta}=\frac{\alpha\theta}{1+\zeta}\sum_{i,j=0}^{\infty}\sum_{\zeta=0}^{l}(-1)^{j+\zeta+l}\binom{-2}{i}\binom{l}{\zeta}\binom{-(i+1)\alpha}{j}\binom{\theta\left[j+\alpha(i+1)\right]-1}{l},$$

and $\pi_{((1+\zeta)c,a,b)}(x)$ is the PDF of the IW model with scale parameter $a\left[(1+\zeta)c\right]^{\frac{1}{b}}$ and shape parameter b. So, the new density (6) can be expressed as a double linear mixture of the IW density. Then, several of its structural properties can be obtained from Equation (8) and those properties of the IW model. By integrating Equation (8), the CDF of X becomes:

$$F_{(\alpha,\theta,c,a,b)}(x)=\sum_{\zeta=0}^{\infty} b_{\zeta}\Pi_{((1+\zeta)c,a,b)}(x), \tag{9}$$

where, $\Pi_{((1+\zeta)c,a,b)}(x)$ is the CDF of the IW distribution with scale parameter $a\left((1+\zeta)c\right)^{\frac{1}{b}}$ and shape parameter b.

Moments and Incomplete Moments

The r^{th} ordinary moment of X is given by:

$$\mu_r' = E\left(X^r\right)=\int_{-\infty}^{\infty} x^r f(x)dx,$$

then we obtain:

$$\mu_r' = \sum_{\zeta=0}^{\infty} b_{\zeta}a^r\left[(1+\zeta)c\right]^{\frac{r}{b}}\Gamma\left(\frac{b-r}{b}\right)\Big|_{[b>r]}, \tag{10}$$

where:

$$\Gamma(1+\kappa)\Big|_{\left(\kappa\in\mathbb{R}^{+}\right)}=\kappa!=\prod_{h=0}^{\kappa-1}(\kappa-h).$$

Setting r = 1, 2, 3 and 4 in (10), we have:

$$E(X)=\mu_1' = \sum_{\zeta=0}^{\infty} b_{\zeta}a\left[(1+\zeta)c\right]^{\frac{1}{b}}\Gamma\left(\frac{b-1}{b}\right),\forall b>1,$$

$$E\left(X^2\right)=\mu_2' = \sum_{\zeta=0}^{\infty} b_{\zeta}a^2\left[(1+\zeta)c\right]^{2/b}\Gamma\left(\frac{b-2}{b}\right),\forall b>2,$$

$$E\left(X^3\right)=\mu_3' = \sum_{\zeta=0}^{\infty} b_{\zeta}a^3\left[(1+\zeta)c\right]^{3/b}\Gamma\left(\frac{b-3}{b}\right),\forall b>3,$$

and:

$$E\left(X^4\right)=\mu_4' = \sum_{\zeta=0}^{\infty} b_{\zeta}a^4\left[(1+\zeta)c\right]^{4/b}\Gamma\left(\frac{b-4}{b}\right),\forall b>4.$$

We can obtain variance Var(X) skewness (Ske(X)) and kurtosis (Ku(X)) measures using the well known relationships.

Numerical Analysis for the E(X), Var(X), Ske(X) and Ku(X) Measures

Numerical analysis for the E(X), Var(X), Ske(X) and Ku(X) are calculated in Table 2 using (10) and well-known relationships for some selected values of parameter α, θ, c, a and b using the *R* software. Based on Table 2 we note that:

1- The skewness of the proposed model is always positive.
2- The kurtosis of the proposed model can be only more than 3.
3- The parameter θ has no effect on the Ske(X) and Ku(X). As illustated in Table 2, skewness = 3.3797 and kurtosis = 74.56 for all values of parameters.
4- The parameter c has no effect on the Ske(X) and Ku(X). As illustated in Table 2, skewness = 2.5307 and kurtosis = 31.1798 for all values of parameters.
5- The parameter a has no effect on the Ske(X) and Ku(X). As illustated in Table 2, skewness = 1.555626 and kurtosis = 9.54016 for all values of parameters.
6- The mean of the proposed model increases as α and b decreases.
7- The mean of the proposed model decreases as θ and a decreases.

The rth incomplete moment, say $\varphi_r(t)$, of X can be expressed, from (8), as:

$$\varphi_r(t) = \int_{-\infty}^{t} x^r f(x)dx = \sum_{\zeta=0}^{\infty} b_\zeta \int_{-\infty}^{t} x^r \pi_{(1+\zeta)c}(x;a,b)dx$$
$$= \sum_{\zeta=0}^{\infty} b_\zeta a^r \left[(1+\zeta)c\right]^{\frac{r}{b}} \gamma\left(\frac{b-r}{b}, \left[(1+\zeta)c\right]\left(\frac{a}{t}\right)^b\right)|_{[b>r]}, \quad (11)$$

where, $\gamma(\kappa, q)$ is the incomplete gamma function, where:

$$\gamma(\kappa,q)|_{(\kappa\neq 0,-1,-2,\ldots)} = \int_0^q t^{\kappa-1}\mathrm{ex}(-t)dt$$
$$= \frac{q^\kappa}{\kappa}\left\{1F_1\left[\kappa;\kappa+1;-q\right]\right\}$$
$$= \sum_{\zeta=0}^{\infty} \frac{(-1)^\zeta}{\zeta!(\kappa+\zeta)} q^{\kappa+\zeta},$$

and ${}_1F_1\,[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. The first incomplete moment given by (11) with $r = 1$ is:

$$\varphi_1(t) = \sum\nolimits_{\zeta=0}^{\infty} w_\zeta a\left[(1+\zeta)c\right]^{\frac{1}{b}}$$
$$\gamma\left(1-\frac{1}{b}, \left[(1+\zeta)c\right]\left(\frac{a}{t}\right)^b\right)|_{[b>1]}.$$

Moment Generating Function (MGF)

The MGF $M_X(t) = E\,(e^{tX})$ of X can be derived from Equation (8) as:

$$M_X(t) = \sum_{\zeta=0}^{\infty} b_\zeta M_{(1+\zeta)c}(t;a,b),$$

where, $M_{(1+\zeta)c}\,(t, a, b)$ is the MGF of the IW model with scale parameter $a\left[(1+\zeta)c\right]^{\frac{1}{b}}$ and shape parameter b, then:

$$M_X(t) = \sum_{\zeta=0}^{\infty}\sum_{r=0}^{\infty}\left(t^r b_\zeta / r!\right)a^r\left[(1+\zeta)c\right]^{\frac{r}{b}}\Gamma\left(\frac{b-r}{b}\right)|_{[b>r]}.$$

We also can determine the generating function of $\pi_{(c\,=\,1,a,b)}(t)$ in (1) by setting $x^{-1} = y$, we can write this MGF as:

$$M(t;a,b) = ba^b \int_0^\infty y^{(b-1)} e^{ty^{-1}} e^{-(ay)^b}.$$

By expanding the first exponential and calculating the integral, we have:

$$M(t;a,b) = ba^b \int_0^\infty \sum_{m=0}^{\infty} \frac{t^m}{m!} y^{b-m-1} e^{ty^{-1}} e^{-(ay)^b}$$
$$= \sum_{m=0}^{\infty} \frac{(at)^m}{m!}\Gamma\left(\frac{b-m}{b}\right).$$

Consider the Wright generalized hypergeometric function (Wright (1935)) defined by:

$${}_p\Psi_q\left[\begin{matrix} a_1, A_1, \ldots, a_p, A_p \\ b_1, B_1, \ldots, b_q, B_q \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma\left(a_j + A_j n\right)x^n}{\prod_{j=1}^{q}\Gamma\left(b_j + B_j n\right)n!}.$$

Then, we can write $M(t, \alpha, \beta)$ as:

$$M(t;a,b) = {}_1\Psi_0\left[\begin{matrix}\left(1,-\frac{1}{b}\right) \\ - \end{matrix}; at\right]. \quad (12)$$

Combining expressions (10) and (12), we obtain the MGF of X, say $M(t)$, as:

$$M(t) = \sum_{\zeta=0}^{\infty} b_\zeta \left\{ {}_1\Psi_0\left[\begin{matrix}\left(1,-\frac{1}{b}\right) \\ - \end{matrix}; a\left[c(\zeta+1)\right]^{\frac{1}{b}} t\right]\right\}.$$

Residual Life and Reversed Residual Life Functions

The n^{th} moment of the residual life is given by:

$$m_n(t) = E\left[(X-t)^n\,|_{X>t,n=1,2,\ldots}\right]$$

Table 2: E(X), Var(X), Ske(X) and Ku(X) of the GOLLIW distribution

α	θ	c	a	b	E(X)	Var(X)	Ske(X)	Ku(X)
100	10.0	1.50	0.5	0.25	1.28887×e^{-110}	6.020703×e^{-108}	1.90377×e^{56}	3.62434×e^{112}
50					1.715037×e^{-53}	8.01146×e^{-51}	5.218939×e^{27}	2.723733×e^{55}
20					1.956096×e^{-19}	9.13752×e^{-17}	4886792644	2.388074×e^{21}
15					8.108882×e^{-14}	3.787905×e^{-11}	75899437	5.760725×e^{15}
3	100.0	1.50	1.25	1.50	48.01294	262.9458	3.379703	74.56242
	50.0				30.24626	104.3501	3.379875	74.56039
	25.0				19.05395	41.41137	3.379702	74.56241
	15.0				13.55457	20.95662	3.379703	74.56242
	10.0				10.34407	12.20486	3.379696	74.56249
	5.0				6.516358	4.843503	3.379703	74.56242
	1.0				2.228563	0.5664996	3.379702	74.5624
	0.5				1.403907	0.2248155	3.379702	74.56241
4	3.0	100.00	1.5	1.25	202.4089	3491.035	2.530743	31.17976
		50.00			116.2534	1151.612	2.530743	31.17976
		25.00			66.77003	379.8904	2.530743	31.17977
		15.00			44.37134	167.7646	2.530743	31.17976
		10.00			32.07964	87.69085	2.530743	31.17977
		5.00			18.42492	28.92719	2.530743	31.17977
		1.00			5.084281	2.202694	2.530743	31.17977
		0.50			2.920153	0.7266182	2.530743	31.17976
		0.10			0.8058042	0.05532918	2.530743	31.17977
		0.05			0.462813	0.01825182	2.530743	31.17977
2	5	3.00	100.0	5.00	188.8523	682.3921	1.555626	9.540161
			50.0		94.42613	170.598	1.555626	9.54016
			25.0		47.21306	42.64951	1.555626	9.540161
			15.0		28.32784	15.35382	1.555626	9.540161
			5.0		9.442613	1.70598	1.555495	9.543932
			1.0		1.888523	0.0682392	1.555626	9.540162
			0.1		0.1888523	0.000683	1.555644	9.53973
4	2	4	3.0	15.0	3.535951	0.005985	0.5553462	4.714781
				10.0	3.839517	0.01596723	0.6110495	4.884068
				5.0	4.919287	0.1068304	0.7869765	5.534967
				4.0	5.570493	0.2163028	0.8808663	5.959772
				3.0	6.857217	0.5940495	1.048379	6.86608
				2.0	10.41556	3.228266	1.440295	9.880603

the n^{th} moment of the residual life of X is given by:

$$m_n(t) = \frac{\int_t^{\infty} (x-t)^n \, dF_{(\alpha,\theta,c,a,b)}(x)}{1-F(t)}.$$

Therefore:

$$m_n(t) = \frac{a^n}{1-F_{(\alpha,\theta,c,a,b)}(t)} \sum_{\zeta=0}^{\infty} v_\zeta \left[(1+\zeta)c\right]^{\frac{n}{b}} \Gamma\left(\frac{b-n}{b}, \left[(1+\zeta)c\right]\left(\frac{a}{t}\right)^b\right)|_{[b>n]},$$

where:

$$v_\zeta = b_\zeta \sum_{r=0}^{n} \binom{n}{r} (-t)^{n-r},$$

$$\Gamma(\kappa,q)|_{x>0} = \int_q^{\infty} t^{\kappa-1} \exp(-t)\, dt,$$

and

$$\Gamma(\kappa,q) + \gamma(\kappa,q) = \Gamma(\kappa).$$

in the upper incomplete gamma function. Another interesting function is the Mean Residual Life (MRL) function or the life expectation at age t defined by $m_1(t) = E[(X-t)|_{X>t,n=1}]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n = 1$ in equation of $m_n(t)$. The n^{th} moment of the reversed residual life, say:

$$M_n(t) = E\left[(t-X)^n |_{X \le t, t>0 \, and \, n=1,2,\ldots}\right]$$

uniquely determines $F(x)$. We obtain:

$$M_n(t) = \frac{\int_0^t (t-x)^n \, dF_{(\alpha,\theta,c,a,b)}(x)}{F_{(\alpha,\theta,c,a,b)}(t)}.$$

Then, the n^{th} moment of the reversed residual life of X becomes:

$$M_n(t) = \frac{a^n}{F_{(\alpha,\theta,c,a,b)}(t)} \sum_{\zeta=0}^{\infty} \eta_\zeta \left[(1+\zeta)c\right]^{\frac{n}{b}}$$
$$\gamma\left(\frac{b-n}{b}, \left[(1+\zeta)c\right]\left(\frac{a}{t}\right)^b\right)|_{[b>n]},$$

where:

$$\eta_\zeta = b_\zeta \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}.$$

Estimation

Let $x_1,\ldots, x_n$ be a random sample from the GOLLIW distribution with parameters α, θ, c, a and b. Let $\Theta = (\alpha, \theta, c, a, b)^T$ be the 5×1 parameter vector. For determining the Maximum Likelihood Estimations (MLEs) of Θ, we have the log-likelihood function:

$$\ell = \ell(\Theta) n \log\left(\alpha\theta c b a^b\right) - (b+1)\sum_{i=1}^{n} \log(x_i) - \alpha\theta c \sum_{i=1}^{n} \left(a x_i^{-1}\right)^b$$
$$+2\sum_{i=1}^{n} \log\left(e^{-\alpha\theta c\left(a x_i^{-1}\right)^b} + \left\{1 - e^{-\theta c\left(a x_i^{-1}\right)^b}\right\}^\alpha\right)$$
$$+(\alpha-1)\sum_{i=1}^{n} \log\left\{1 - e^{-\theta c\left(a x_i^{-1}\right)^b}\right\}.$$

The components of the score vector, $U(\Theta) = \frac{\partial \ell}{\partial \Theta} = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial c}, \frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}\right)^T$, are available if needed. Setting $U_\alpha = U_\theta = U_c = U_a =$ and $U_b = 0$ and solving them simultaneously yields the MLE $\hat{\Theta} = \left(\hat{\alpha}, \hat{\theta}, \hat{c}, \hat{a}, \hat{b}\right)^T$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the 5×5 observed information matrix:

$$J(\Theta) = \left\{\partial^2 \ell / \partial r \partial s\right\}(\forall r, s = \alpha, \theta, c, a, b),$$

Whose elements can be computed numerically.

Simulation Studies

Using Equation (7), we simulate the proposed model by taking $n = 50$, $n = 100$, $n = 200$, $n = 500$ and $n = 1000$ and some selected parameters values:

- **I:** $\alpha = 10.5$, $\theta = 1$, $c = 2$, $a = 0.1$ and $b = 2.1$
- **II:** $\alpha = 1$, $\theta =1$, $c =3$, $a = 0.1$ and $b = 1$
- **III:** $\alpha = 2$, $\theta = 5$, $c = 1$, $a = 1$ and $b = 1$
- **IV:** $\alpha = 3$, $\theta = 3$, $c =4$, $a = 1$ and $b = 1.5$
- **V:** $\alpha = 1.5$, $\theta = 2.5$, $c = 5$, $a = 1.5$ and $b = 1.5$
- **VI:** $\alpha = 8$, $\theta = 3$, $c = 1$, $a = 4$ and $b = 5$
- **VII:** $\alpha = 3$, $\theta = 2$, $c = 3$, $a = 2$ and $b = 4$

For each sample size, we evaluate the sample means and Mean Squared Error (MSE) using the *R* software (optim function). Then, we repeat this process 1000 times. From Table 3, we note that the empirical means approach to the true parameter values when the sample size n increases. The MSEs decrease when the sample size n increases as expected. The results given in Table 3 are in agreement with first-order asymptotic theory.

Real Data Modeling

In this section, we provide three applications to real data sets to illustrate the importance of the GOLLIW distribution. To evaluate performance of considered model, the MLEs of the parameters for the considered models are calculated and three goodness-of-fit statistics are used to compare the new distribution. The measures of goodness of fit including anderson-Darling ($A^\star$), Cramérvon Mises ($W^\star$) and Kolmogrov-Smirnov (K-S) statistics are computed to compare the fitted models. The statistics $A^\star$ and $W^\star$ are described in details in Chen and Balakrishnan (1995). In general, the smaller are the values of these statistics, the better the fit to the data. The required computations are carried out in the R-language for the first three application. The numerical values of the model selection statistics $A^\star$, $W^\star$ and K-S are listed in Tables 4, 6 and 8. Tables 5, 7 and 9 list the MLEs and their corresponding standard errors (in parentheses) of the model parameters. The Total Time Test (TTT) plots for the three data set indicates that the HRFs are increasing, increacing and increasing.

The statistics of the fitted models for the 1st data set are presented in Table 4 and the MLEs and corresponding standard errors are given in Table 5. We note from the figures in Table 4 that the GOLLIW model has the lowest values of the $A^\star$, $W^\star$ and K-S statistics (for the 1st data set) as compared to their submodels, suggesting that the GOLLIW model provide the best fit.

The histogram of the 1st data and other important plots are displayed in Fig. 2.

We compare the fits of the GOLLIW distribution with other models such as Inverse Weibull (IW), Kumaraswamy Inverse Weibull (KIW), exponentiated Inverse Weibull (EIW), beta Inverse Weibull (BIW), transmuted Inverse Weibull (TIW), Marshal-Olkin Inverse Weibull (MOIW) and McDonald Inverse Weibull (McIW) distributions given by:

EIW:

$$f_{EIW}\left(x;\alpha,a,b\right)=\alpha b a^{b} x^{-(b+1)} e^{-\left(ax^{-1}\right)^{b}}\left\{1-e^{-\left(ax^{-1}\right)^{b}}\right\}^{\alpha-1};$$

BIW:

$$f_{BIW}\left(x;\alpha,c,a,b\right)=b a^{b} B^{-1}\left(\alpha,c\right) x^{-(b+1)} e^{-\alpha\left(ax^{-1}\right)^{b}}\left\{1-e^{-\left(ax^{-1}\right)^{b}}\right\}^{c-1};$$

KIW:

$$f_{KIW}\left(x;\alpha,c,a,b\right)=\alpha c b a^{b} x^{-(b+1)} e^{-\alpha\left(ax^{-1}\right)^{b}}\left\{1-e^{-a\left(ax^{-1}\right)^{b}}\right\}^{c-1};$$

TIW:

$$f_{TIW}\left(x;\alpha,a,b\right)=b a^{b} x^{-(b+1)} e^{-\left(ax^{-1}\right)^{b}}\left\{1+\alpha-2\alpha e^{-\left(ax^{-1}\right)^{b}}\right\};$$

MOIW:

$$f_{MOIW}\left(x;\alpha,a,b\right)=\alpha b a^{b} x^{-(b+1)} e^{-\left(ax^{-1}\right)^{b}}\left\{\alpha+\left(1-\alpha\right)e^{-\left(ax^{-1}\right)^{b}}\right\}^{-2};$$

McIW:

$$f_{McIW}\left(x;\alpha,c,\lambda,a,b\right)=\lambda b a^{b} x^{-(b+1)} B^{-1}\left(\alpha,c\right) e^{-\left(ax^{-1}\right)^{b}}\left(e^{-\left(ax^{-1}\right)^{b}}\right)^{\alpha\lambda-1}$$
$$\times\left(1-\left(e^{-\left(ax^{-1}\right)^{b}}\right)^{\lambda}\right)^{c-1}.$$

Table 3: Sample means (MSE) for the GOLLIW distribution

Parameters	$\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$	$\hat{a}$	$\hat{b}$
n = 50					
I	10.7892(0.3679)	1.0892(0.39182)	1.79820(0.4901)	0.0795(0.3907)	2.0178(0.289)
II	1.1797(0.6699)	1.1537(0.4898)	2.80898(0.4861)	0.0028(0.4845)	1.1099(0.49108)
III	1.7589(0.3911)	4.7604(0.2943)	0.79164(1.0044)	0.9499(0.1971)	0.9791(0.4609)
1V	4.8983(0.3475)	3.0811(0.5660)	3.82989(0.3954)	1.0235(0.2949)	1.38901(0.3720)
V	1.3478(0.5098)	2.3989(0.7279)	4.66210(0.9951)	1.7676(0.6054)	1.5560(0.5619)
VI	7.8260(0.7912)	2.9546(0.8555)	0.78114(0.9173)	5.2145(1.6780)	4.7813(1.1468)
VII	2.6183(0.7909)	2.1863(0.8339)	3.78962(0.35030)	1.8569(0.9576)	4.14751(0.8011)
n = 100					
I	10.6916(0.3368)	1.2298(0.2912)	1.80843(0.3688)	0.0499(029298)	2.0889(0.2666)
II	1.0501(0.5014)	1.0143(0.3501)	2.61621(0.3799)	0.0570(0.3129)	0.5065(0.3189)
III	1.8991(0.2649)	5.0912(0.3481)	0.83867(0.8978)	1.0705(0.6892)	1.1379(0.1319)
1V	2.6639(0.1613)	3.0291(0.5126)	3.89781(0.1898)	0.9589(0.191)	1.5389(0.1131)
V	1.5310(0.5106)	2.5299(0.6039)	4.80391(0.6609)	1.2505(0.5031)	1.4944(0.391)
VI	6.7800(0.3134)	2.9040(0.5019)	0.84342(0.60559)	5.0641(0.4601)	4.9110(0.6615)
VII	2.9606(0.6047)	2.1078(0.6173)	3.30520(0.23989)	1.9690(0.5806)	4.6582(0.6102)
n = 200					
I	10.5586(0.0120)	0.9847(0.0170)	1.89676(0.21767)	0.0958(0.01982)	1.98903(0.0069)
II	1.0123(0.054)	1.0608(0.0270)	2.82982(0.2390)	0.0106(0.0099)	1.050(0.1005)
III	2.000(0.02395)	5.0005(0.1479)	0.90120(0.6179)	0.9792(0.1028)	0.992(0.078)
1V	2.994(0.1907)	2.0129(0.026)	3.92340(0.11989)	0.992(0.019)	1.506(0.0992)
V	1.4994(0.347)	2.4950(0.402)	4.86397(0.4556)	1.4958(0.346)	1.4960(0.257)
VI	7.9912(0.0349)	2.9701(0.0260)	0.89589(0.41506)	5.0098(0.0607)	4.9979(0.0895)
VII	2.8946(0.0399)	2.0968(0.0527)	3.25576(0.17900)	1.9607(0.0898)	4.0020(0.0300)
n = 500					
I	10.5028(0.0039)	0.99941(0.0085)	1.99998(0.13343)	0.10119(0.0054)	1.9891(0.0061)
II	1.0020(0.00231)	1.00701(0.00411)	2.98341(0.11090)	0.10829(0.0083)	1.024(0.0031)
III	2.0039(0.00960)	5.0017(0.00701)	0.96189(0.2009)	1.00678(0.0081)	0.9999(0.0022)
1V	2.9999(0.00091)	3.0046(0.00402)	3.99999(0.0020)	2.00848(0.0011)	1.5029(0.0090)
V	1.4991(0.00851)	2.55841(0.0013)	4.9489(0.2011)	1.49986(0.0041)	1.4964(0.00190)
VI	7.9929(0.2015)	2.99647(0.2962)	0.98660(0.11003)	5.00979(0.4785)	4.9894(0.458)
VII	2.90367(0.3165)	2.00539(0.3479)	3.110761(0.0043)	2.00438(0.3150)	4.000989(0.3433)
n = 1000					
I	10.50015(0.0015)	0.9991(0.0980)	2.0007(0.001734)	0.1000(0.0004)	2.01012(0.089)
II	1.00011(0.0070)	1.0002(0.0979)	2.9992(0.00953)	0.10061(0.0024)	1.00081(0.090)
III	1.9999(0.0667)	5.00007(0.0373)	0.9965(0.00702)	0.9989(0.0315)	1.0010(0.028)
1V	3.0001(0.0139)	3.0015(0.1052)	4.00080(0.000817)	0.9999(0.0409)	1.50004(0.031)
V	1.4998(0.00089)	2.4988(0.00910)	4.9999(0.00542)	1.5001(0.0014)	1.4998(0.127)
VI	8.00034(0.00001)	3.0008(0.00091)	0.9998(0.010003)	4.9997(0.00047)	5.0016(0.0007)
VII	3.0004(0.00020)	2.0003(0.00113)	3.0012(0.00001)	2.0005(0.0033)	5.0065(0.0001)

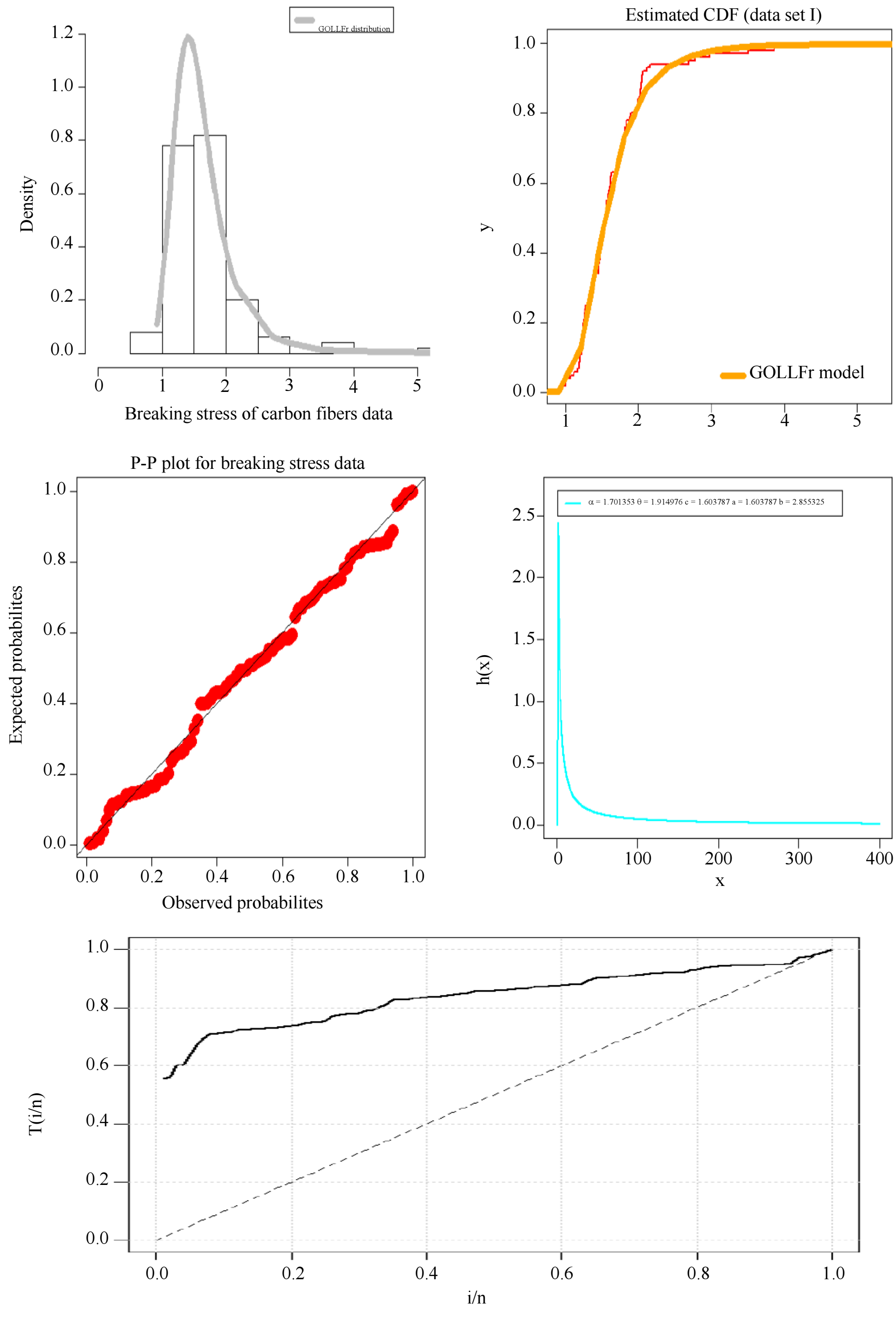

Fig. 2: Estimated PDF, estimated CDF, P-P plot, estimated HRF and TTT plot for data set I

The parameters of the above densities are all positive real numbers except for the TIW distribution for which $|\alpha|\leq 1$.

Breaking Stress Data

The 1st data set is an uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba) given by Nichols and Padgett (2006) and these data are used by Mahmoud and Mandouh (2013) to fit the transmuted IW distribution. The statistics of the fitted models are presented in Table 4 and the MLEs and corresponding standard errors are given in Table 5. We note from Table 4 that the GOLLIW gives the lowest values the $A^{\star}$, $W^{\star}$ and

K-S statistics (for the 1st data set) as compared to further models and therefore the new one can be chosen as the best one. The histogram of the 1st data and other important plots are displayed in Fig. 3.

Glass Fibers Data

The 2nd data set is generated data to simulate the strengths of glass fibers which was given by Smith and Naylor (1987).

The statistics of the fitted models are presented in Table 6 and the MLEs and corresponding standard errors are given in Table 7. It is clear from Table 6 that the GOLLIW gives the lowest values the $A^\star$, $W^\star$ and K-S statistics (for the 2nd data set) as compared to other models and therefore our new model can be chosen as the best one. The histogram of the 2nd data are displayed in Fig. 3. We compare the fits of the GOLLIW distribution with other IW models such as KIW, BIW, EIW, IW, TIW, MOIW and McIW.

Table 4: $A^\star$, $W^\star$, K-S and K-S p-value for data set I

	Goodness of fit criteria			
Model	$W^\star$	$A^\star$	K-S	p-value
GOLLIW	0.06234	0.4835	0.0637	0.8117
GOLLIW	0.06238	0.4837	0.0638	0.8101
OLLEIW	0.1203	0.9639	0.5561	$2.2\times e^{-16}$
OLLEIR	0.1553	1.21197	0.65497	$2.2\times e^{-16}$
OLLIR	0.15532	1.21201	0.6550	$2.2\times e^{-16}$
IW	0.1090	0.7657	0.0874	0.4282
KIW	0.0812	0.6217	0.0759	0.6118
EIW	0.1091	0.7658	0.0874	0.4287
BIW	0.0809	0.6207	0.0757	0.6147
TIW	0.0871	0.6209	0.0782	0.5734
MOIW	0.0886	0.6142	0.0763	0.5168
McIW	0.1333	1.0608	0.0807	0.5332

Table 5: MLEs and their standard errors (in parentheses) for data set I

	Estimates				
Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$	$\hat{a}$	$\hat{b}$
GOLLIW	1.7014	1.9149	1.6038	0.91205	2.8553
	(0.66)	(20.09)	(16.78)	(4.035)	(1.0096)
GOLLIW	1.6989	2.5203		0.9779	2.8584
	(0.655)	(27.6)		(3.75)	(1.0037)
OLLEIW	0.1351		3.7216	0.9296	21.319
	(0.011)		(0.0034)	(0.0033)	(0.0034)
OLLEIR	0.4946		0.067	1.74262	2
	(0.04135)		(0.7195)	(9.3007)	
OLLIR	0.49459			0.45242	2
	0.04135			0.03869	
IW				1.3968	4.3724
				(0.0336)	(0.3278)
KIW		0.8489	1.6239	1.6341	3.4208
		(16.083)	(0.6979)	(9.049)	(0.7635)
EIW		0.9395		1.4169	0.9395
		(3.543)		(2.568)	(0.3278)
BIW		0.7346	1.5830	1.6684	3.5112
		(1.5290)	(0.7132)	(0.7662)	(0.9683)
TIW	-0.7166			1.2656	4.7121
	(0.2616)			(0.0579)	(0.3657)
MOIW		0.0033		6.2296	1.2419
		(0.0009)		(1.0134)	(0.1181)
McIW	0.8503	44.423	19.859	0.0203	46.974
	(0.1353)	(25.100)	(6.706)	(0.0060)	(21.871)

DOI: 10.3844/jmssp.2019.30.43

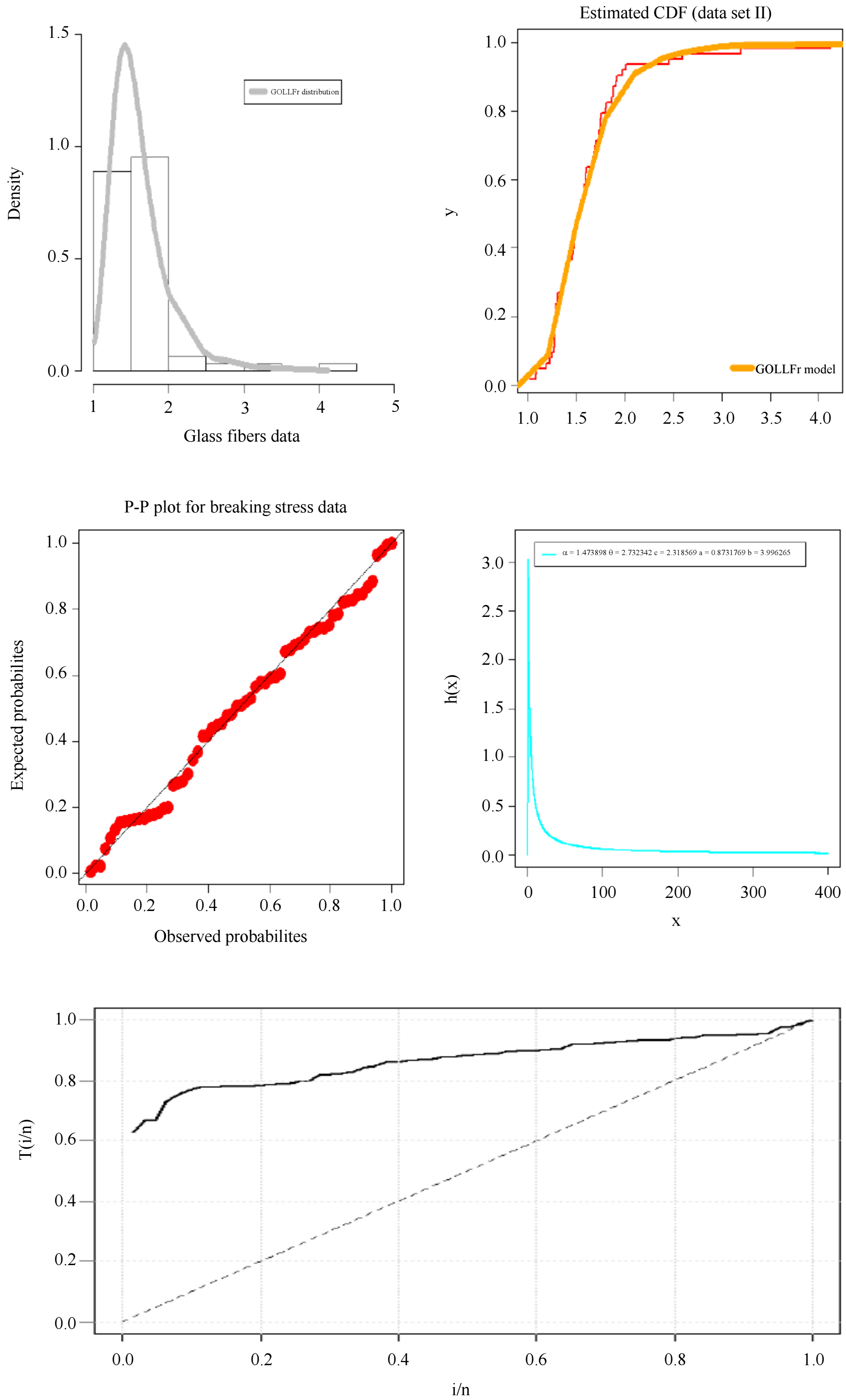

Fig. 3: Estimated PDF, estimated CDF, P-P plot, estimated HRF and TTT plot for data set II

Table 6: $A^\star$, $W^\star$, K-S and p-value for data set II

	Goodness of fit criteria			
Model	$W^\star$	$A^\star$	K-S	p-value
GOLLIW	0.05082	0.4115	0.0690	0.9043
OLLEIW	0.05084	0.4117	0.06901	0.9040
GOLLIR	0.05250	0.4530	0.070334	0.8926
OLLEIW	0.10487	0.8325	0.55196	$6.661\times e^{-16}$
OLLEIR	0.1502	1.14697	0.67949	$6.661\times e^{-16}$
OLLIR	0.15021	1.14697	0.67951	$6.661\times e^{-16}$
IW	0.0707	0.5332	0.0772	0.8185
KIW	0.0634	0.4981	0.0715	0.8810
EIW	0.0707	0.5332	0.0772	0.8187
BIW	0.0640	0.5008	0.0716	0.8804
TIW	0.0655	0.4939	0.0735	0.8470
MOIW	0.0629	0.4902	0.0813	0.7685
McIW	0.1161	0.9193	0.0831	0.7455

Table 7: MLEs and their standard errors (in parentheses) for data set II

	Estimates				
Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$	$\hat{a}$	$\hat{b}$
GOLLIW	1.4739	2.7323	2.3186	0.8732	3.9963
	(0.629)	(0.000)	(0.000)	(0.000)	(1.5198)
OLLEIW	0.1449		0.00879	1.2997	24.878
	(0.0129)		(0.000)	(0.000)	(0.000)
GOLLIR	3.032	2.1568		0.86599	2
	(0.325)	(9.145)		(1.836)	
OLLEIR	0.5025		0.0716	1.7048	2
	(0.0529)		(1.13062)	(13.47)	
OLLIR	0.50251			0.45599	2
	0.052946			0.048652	
IW				1.4108	5.4377
				(0.0344)	(0.5192)
KIW		0.2855	1.2824	1.9142	4.7731
		(9.1338)	(0.6388)	(12.836)	(1.3134)
EIW		0.9059		1.4367	5.4379
		(2.764)		(4.324)	(0.5193)
BIW		1.2996	1.2649	1.3945	4.7927
		(4.4378)	(0.6640)	(0.9304)	(1.4641)
TIW	0.7778			1.5491	4.3139
	(0.2477)			(0.0655)	(0.5849)
MOIW		0.0023		5.2383	1.4537
		(0.0004)		(0.8209)	(0.1650)
McIW		56.227	14.953	0.0073	29.104
		(30.539)	(4.733)	(0.0013)	(11.304)

Table 8: $A^\star$, $W^\star$, K-S and K-S p-value for data set III

Model	$W^\star$	$A^\star$	K-S	p-value
GOLLIW	0.1195	0.9129	0.10406	0.6511
OLLIW	0.15615	1.1276	0.10476	0.6427
GOLLIR	0.19551	1.3498	0.11008	0.5797
OLLEIW	0.1577	1.09876	0.53498	$7.436\times e^{-13}$
OLLEIR	0.14258	1.02646	0.62685	$2.2\times e^{-16}$
OLLIR	0.14257	1.0264	0.62687	$2.2\times e^{-16}$
IW	0.3233	2.0301	0.1506	0.2066
EIW	0.3233	2.0301	0.1506	0.2064
TIW	0.2823	1.8152	0.1370	0.3045

Relief Time Data

The 3rd data set (Wingo data) represents a complete sample from a clinical trial describe a relief time (in hours) for 50 arthritic patients (Wingo (1983)).

We will compare the fits of the GOLLIW distribution with other models such as IW, EIW and TIW. The statistics of the fitted models are presented in Table 8 and the MLEs and corresponding standard errors are given in Table 9. It is clear from Table 8 that the GOLLIW gives the lowest values of the $A^{\star}$, $W^{\star}$ and K-S statistics (for the 3rd data set) as compared to other models and therefore the new model can be chosen as the best one. The histogram of the 3rd data ar displayed in Fig. 4.

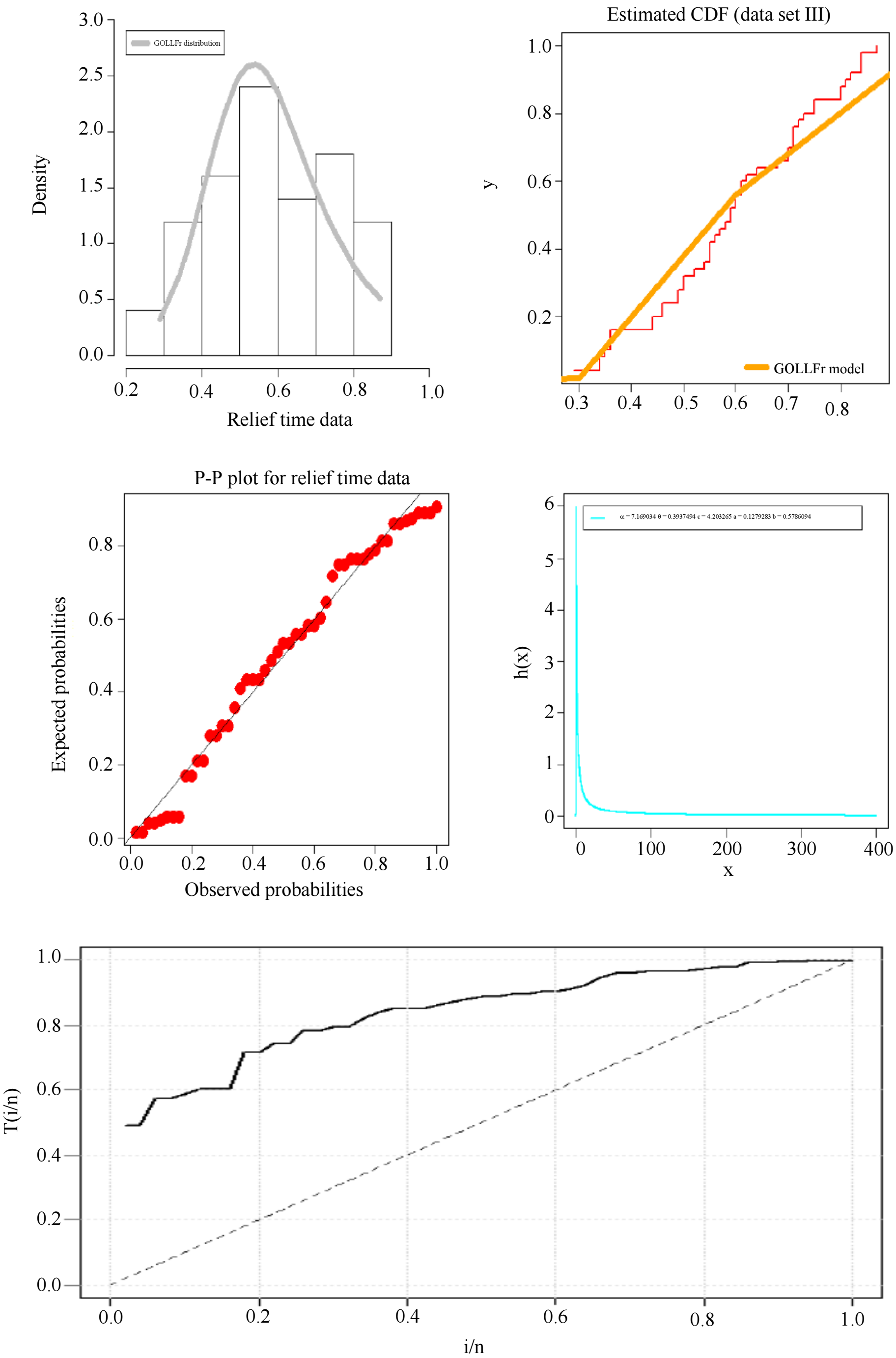

Fig. 4: Estimated PDF, estimated CDF, P-P plot, estimated HRF and TTT plot for data set III

Table 9: MLEs and their standard errors (in parentheses) for data set III

Model	Estimates $\hat{\alpha}$	$\hat{\theta}$	$\hat{c}$	$\hat{a}$	$\hat{b}$
GOLLIW	7.1690	0.3937	4.2033	0.12793	0.5786
	(0.000)	(0.1430)	(1.5104)	(0.000)	(0.000)
OLLIW	2.899		0.09875	2.4072	1.34989
	(0.8263)		(0.05666)	(0.8460)	(0.34177)
GOLLIR	1.961	0.111		1.4123	2
	(0.234)	(0.000)		(0.000)	
OLLEIW	0.0669		0.00459	0.3558	32.561
	(0.0076)		(0.0028)	(0.0047)	(0.006)
OLLEIR	0.48388		0.631294	0.198037	2
	(0.05702)		(0.000)	(0.000)	
OLLIR	0.4837			0.15732	2
	(0.057)			(0.0193)	
IW				0.4859	3.2078
				(0.0227)	(0.3263)
EIW			0.9047	0.5013	3.2077
			(18.784)	(3.2444)	(0.3263)
TIW	-0.5816			0.4400	3.4974
	(0.2787)			(0.0290)	(0.3527)

Concluding Remarks

We introduce a new flexible extension of the Inverse Weibull distribution. Some important mathematical properties of the proposed model are derived along with a numerical analysis of mean, variance, skewness and kurtosis measures of the proposed model. The performance of the maximum likelihood method is assesses via a comprehensive simulation studies in terms of mean squared errors. The new model is better than some other important competitive extensions of the Inverse Weibull in modeling the breaking stress data, the glass fibers data and the relief time data. Some plots such as estimated PDF, estimated CDF, P-P plot, estimated HRF and TTT are given to illustrate the suitability of the new model to fit the used data sets.

Based the numerical analysis of mean, variance, skewness and kurtosis measures we summarize the following concluding remakes:

1- The skewness of the proposed model is always positive.
2- The kurtosis of the proposed can be less than 3 and more than 3
3- The parameter θ has no effect on the skewness and kurtosis: As illustated in Table 2, skewness = 3.3797 and kurtosis = 74.56 for all values of parameters
4- The parameter c has no effect on the Ske(X) and kurtosis: As illustated in Table 2, skewness = 2.5307 and kurtosis = 31.1798 for all values of parameters
5- The mean of the proposed model increases as α and b decreases
6- The mean of the proposed model decreases as θ and a decreases

Acknowledgment

The author gratefully acknowledge with thanks the very thoughtful and constructive suggestions and comments of the Editor-in-Chief and the reviewers which resulted in much improved paper.

Author's Contributions

The authors jointly written and developed all the paper together.

Ethics

The author declares that there is no conflict of interests regarding the publication of this article.

References

Afify, A.Z., G.G. Hamedani, I. Ghosh and M.E. Mead, 2015. The transmuted Marshall. Olkin Fréchet distribution: Properties and applications. Int. J. Stat. Probability, 4: 132-184.

Barreto-Souza, W.M., G.M. Cordeiro and A.B. Simas, 2011. Some results for beta Fréchet distribution. Commun. Stat.-Theory Meth., 40: 798-811. DOI: 10.1080/03610920903366149

Cordeiro, G.M., M.M. Alizadeh, G. Ozel, B. Hosseini and E.M.M. Ortega *et al*., 2016. The generalized odd log-logistic family of distributions: Properties, regression models and applications. J. Stat. Comput. Simulat., 87: 908-932.

Fréchet, M., 1927. Sur la loi de probabilité de lécart maximum. Ann. de la Soc. Polonaisede Math., 6: 93-116.

Gusmao, F.R.S., E.M.M. Ortega and G.M. Cordeiro, 2011. The generalized inverse Weibull distribution. Stat. Papers, 52: 591-619. DOI: 10.1007/s00362-009-0271-3

Keller, A.Z. and A.R. Kamath, 1982. Reliability analysis of CNC machine tools. Reliab. Eng., 3: 449-473. DOI: 10.1016/0143-8174(82)90036-1

Krishna, E., K.K. Jose, T. Alice and M.M. Risti, 2013. The Marshall-Olkin Fréchet distribution. Commun. Stat.-Theory Meth., 42: 4091-4107. DOI: 10.1080/03610926.2011.648785

Mahmoud, M.R. and R.M. Mandouh, 2013. On the transmuted Fréchet distribution. J. Applied Sci. Res., 9: 5553-5561.

Mead, M.E., A.Z. Afify, G.G. Hamedani and I. Ghosh, 2017. The beta exponential Fréchet distribution with applications. Austr. J. Stat., 46: 41-63. DOI: 10.17713/ajs.v46i1.144

Nadarajah, S. and S. Kotz, 2003. The exponentiated Fréchet distribution. Interstat Electron. J., 1-7.

Nichols, M.D. and W.J. Padgett, 2006. A bootstrap control chart for Weibull percentiles. Quality Reliability Eng. Int., 22: 141-151. DOI: 10.1002/qre.691

Smith, R.L. and J.C. Naylor, 1987. A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Applied Stat., 36: 358-369. DOI: 10.2307/2347795

Treyer, V.N., 1964. Doklady Acad, Nauk, Belorus, U.S.S.R.

Wingo, D.R., 1983. Maximum likelihood methods for fitting the Burr type XII distribution to life test data. Biometr. J., 25: 77.84.

Wright, E.M., 1935. The asymptotic expansion of the generalized hypergeometric function. J. London Math. Society, 10: 286-293. DOI: 10.1112/jlms/s1-10.40.286

Yousof, H.M., E. Altun and G.G. Hamedani, 2018. A new extension of Frechet distribution with regression models, residual analysis and characterizations. J. Data Sci., 16: 743-770.